A STUDY IN THE GROWTH PROPERTIES AND COEFFICIENTS OF ANALYTIC FUNCTIONS

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BY
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to the

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AUGUST 1972

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To my brother

Dr. O.P. KAPOOR
M.D.

POST GRADUATE OFFICE
Submitted on 21/8/72

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This thesis has been approved for the award of the Degree of Doctor of Philosophy (Ph.D.) in accordance with the regulations of the Indian Institute of Technology Kanpur
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CERTIFICATE

This is to certify that the work embodied in the thesis entitled *'A study in the growth properties and coefficients of analytic functions'* being submitted by *Govind Prakash Kapoor* has been carried out under my supervision and that it has not been submitted elsewhere for the award of any degree or diploma.

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CHAPTER 1

INTRODUCTION

1.1. The growth of analytic functions, as measured by their maximum modulii, plays a very important role in the study of their asymptotic values, exceptional values, distribution of zeros etc. It is also used in the theory of approximation and Differential equations. However, the growth of such functions has been studied in some detail only for a sub-class of analytic functions, namely the functions which are analytic in every finite region of the complex plane. Such functions are called entire or integral functions. The general theory of entire functions originated in the works of Weirstrass [94] and was developed further by Picard [50] , Borel [14] , Laguerre, Hadamard and others, but it seems that first significant step in the direction of the study of their growth was taken by Borel. Many new concepts in the theory were introduced in the beginning of this century by eminent mathematicians such as Valiron, Lindelöf, Nevanlinna etc. Since then the theory has been enriched by the works of Hayman, Boas, Shah, Clunie, Anderson and others.

As compared to the case of entire functions much less work has been done to study the growth of analytic functions in general. Beuermann [10] studied the growth of an analytic function in the unit disc. In the recent years some work concerning the growth of such functions has been done by Tsuji, Linden, MacLane, Sons and others.

1.2. Let $f(z)$ be an entire function of the complex variable $z = re^{i\theta}$ and let

$$M(r) \equiv M(r,f) = \max_{|z|=r} |f(z)|.$$

$M(r)$ is said to be the maximum modulus of $f(z)$ for $|z| \leq r$. Since the maximum is attained on the boundary we write $|z| = r$ instead of $|z| \leq r$. Blumenthal [11] showed that $M(r)$ is a steadily increasing continuous function of r and that it is differentiable in adjacent intervals. Further, by Hadamard's three-circles theorem, it follows that $\log M(r)$ is a convex function of $\log r$ and hence it has the representation

$$(1.2.1) \qquad \log M(r) = \log M(r_0) + \int_{r_0}^{r} \frac{W(x)}{x}\, dx\ ,\ r > r_0,$$

where $W(x)$ is a nonnegative, indefinitely increasing function which is continuous in adjacent intervals.

The entire function $f(z)$ is said to be of finite order if there exists a constant A such that

$$M(r) < \exp\ (r^A)$$

for all sufficiently large values of r.

The greatest lower bound ρ of all such numbers A is called the order of the function $f(z)$. Thus,

$$(1.2.2) \qquad \rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}\ .$$

If no such constant can be found then $f(z)$ is said to be of infinite order. A constant is of zero order by convention.

For a more precise specification of the rate of growth of $f(z)$ the concept of type has been introduced. Thus, the entire function $f(z)$, of nonzero finite order ρ, is said to be of type T if

(1.2.3) $$\limsup_{r \to \infty} \frac{\log M(r)}{r^{\rho}} = T \quad , (0 \leq T \leq \infty).$$

According as $T = \infty$, $0 < T < \infty$ or $T = 0$, $f(z)$ is said to be of maximal, mean or minimal type of order ρ.

Whittaker [95] introduced the concept of lower order of an entire function. Thus, an entire function $f(z)$ is said to be of lower order λ if

(1.2.4) $$\liminf_{r \to \infty} \frac{\log \log M(r)}{\log r} = \lambda \quad , (0 \leq \lambda \leq \rho \leq \infty).$$

In analogy with the lower order, the lower type t of an entire function $f(z)$ of order $\rho \, (0 < \rho < \infty)$ is defined [69] as

(1.2.5) $$\liminf_{r \to \infty} \frac{\log M(r)}{r^{\rho}} = t \, , (0 \leq t \leq T \leq \infty).$$

An entire function $f(z)$ is said to be of regular growth if $\rho = \lambda$ and is said to be of irregular growth if $\rho > \lambda$. A function $f(z)$ of regular growth is said to be of perfectly regular growth if $T = t < \infty$. An entire function $f(z)$ is said to have growth $\{\rho , T\}$ if its order does not exceed ρ and its type does not exceed T if it is of order ρ. An entire function of growth $\{1,T\}$, $T < \infty$ is called a function of exponential type.

Since for functions of irregular growth, the lower type is zero [84] another concept, that of λ-type, may be used for such functions. Thus, an entire function $f(z)$, of order ρ and lower order $\lambda (0 < \lambda < \rho < \infty$ is said to be of λ-type t_λ if

$$(1.2.6) \qquad \liminf_{r\to\infty} \frac{\log M(r)}{r^{\lambda}} = t_{\lambda} .$$

1.3. Since the entire function $f(z)$ is regular everywhere in the finite complex plane, it can be expanded in a Taylor series around any point $z=z_0$ of the plane. However, without loss of generality, we may take $z_0 = 0$. Thus $f(z)$ has the representation

$$(1.3.1) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where the coefficients a_n's are given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} [f(z)/z^{n+1}]dz = \frac{f^n(0)}{n!}$$

$f^n(0)$ being the value of the n^{th} derivative of $f(z)$ at $z = 0$.

Since the sequence $\{a_n\}$ determines the function $f(z)$ completely, theoretically, it should be possible to discover all the properties of the function by examining its coefficients. For the order, lower order, and type this has been done.

Thus, the entire function $f(z)$, defined by (1.3.1), is of finite order, if and only if [13, pp. 9-12] ,

$$(1.3.2) \qquad \limsup_{n\to\infty} \frac{n \log n}{\log |a_n|^{-1}} = \mu \text{ is finite} ;$$

and then the order ρ of $f(z)$ is equal to μ .

For an entire function of infinite order this condition is necessary but not sufficient.

Further, the entire function $f(z)$, defined by (1.3.1), is of order $\rho\ (0 < \rho < \infty)$ and type $T\ (0 < T < \infty)$, if and only if,

$$\limsup_{n\to\infty} n|a_n|^{\rho/n} = e\rho T. \tag{1.3.3}$$

The formulae analogous to (1.3.2) and (1.3.3) do not hold for the lower order and lower type of $f(z)$, i.e., there exist entire functions having lower order λ and lower type t for which the relations

$$\lambda = \liminf_{n\to\infty} \frac{n \log n}{\log |a_n|^{-1}}$$

and

$$e\rho t = \liminf_{n\to\infty} n\, |a_n|^{\rho/n}$$

do not hold. Shah [62,63] proved that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of lower order $\lambda (0 \leq \lambda \leq \infty)$, then

$$\lambda \geq \liminf_{n\to\infty} \frac{n \log n}{\log |a_n|^{-1}} \geq \liminf_{n\to\infty} \frac{\log n}{\log |a_n/a_{n+1}|} \tag{1.3.4}$$

and

$$\rho \leq \limsup_{n\to\infty} \frac{\log n}{\log |a_n/a_{n+1}|}. \tag{1.3.5}$$

Further, if

(1.3.6) $\phi(n) \equiv |a_n/a_{n+1}|$ is a nondecreasing function of n for $n > n_0$,

then

$$\lambda = \liminf_{n\to\infty} \frac{n \log n}{\log |a_n|^{-1}} = \liminf_{n\to\infty} \frac{\log n}{\log |a_n/a_{n+1}|} \tag{1.3.7}$$

and

(1.3.8) $$\rho = \limsup_{n \to \infty} \frac{\log n}{\log |a_n/a_{n+1}|} .$$

Similarly for the entire function $f(z)$, given by (1.3.1), having order $\rho\,(0 < \rho < \infty)$ and lower type $t\ (0 \leq t \leq \infty)$, it has been shown [69] that

(1.3.9) $$e\rho t \geq \liminf_{n \to \infty} n\, |a_n|^{\rho/n}$$

and, further, if (1.3.6) holds, then

(1.3.10) $$e\rho t = \liminf_{n \to \infty} n\, |a_n|^{\rho/n} .$$

Juneja [33] has shown that if $f(z)$, given by (1.3.1), is of order $\rho(\,0 < \rho < \infty)$, type $T(0 \leq T \leq \infty)$ and lower type $t(0 \leq t \leq \infty)$, then

(1.3.11) $$\liminf_{n \to \infty} \frac{n}{\rho} |a_{n+1}/a_n|^{\rho} \leq t \leq T \leq \limsup_{n \to \infty} \frac{n}{\rho} |a_{n+1}/a_n|^{\rho} .$$

Hence, if $\lim_{n\to\infty} n \;\; |a_{n+1}/a_n|^{\rho} = K(0 < \rho, K < \infty)$, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of perfectly regular growth, order ρ and type K/ρ .

Further, if (1.3.6) holds, then

(1.3.12) $$T \leq \limsup_{n \to \infty} \frac{n}{\rho} |a_{n+1}/a_n|^{\rho} \leq e\, T.$$

The inequalities in (1.3.12) are sharp.

Results analogous to (1.3.9) and (1.3.10) for the λ-type of the function $f(z)$ have been obtained by Srivastava and Singh [84] .

A coefficient formula for the lower order, which holds for every entire function, has been obtained independently by Roux [58] and Juneja [34] by using different techniques. Thus, it has been shown that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function then the lower order $\lambda(0 \leq \lambda \leq \infty)$ of $f(z)$ is given by

$$(1.3.13) \qquad \lambda = \max_{\{n_k\}} \liminf_{k \to \infty} \frac{n_k \log n_{k-1}}{\log |a_{n_k}|^{-1}}$$

where maximum is taken over all increasing sequences $\{n_k\}$ of natural numbers.

Recently, another coefficient formula which also gives lower order for every entire function has been obtained by Juneja and Kapoor [35]. Thus it has been shown that if $f(z)$, given by (1.3.1), is of lower order $\lambda(0 \leq \lambda \leq \infty)$, then

$$(1.3.14) \qquad \lambda = \max_{\{n_k\}} \liminf_{k \to \infty} \frac{(n_k - n_{k-1}) \log n_{k-1}}{\log |a_{n_{k-1}}/a_{n_k}|}.$$

Some more results which depict the influence of the coefficients on the growth of entire functions may be found in Clunie [18], R.S.L. Srivastava [81] and S.N. Srivastava [85].

1.4. If $f(z)$ is an entire function of infinite or zero order, the usual definitions of type and lower type are not feasible and so growth of $f(z)$ cannot be precisely measured by confining to these concepts. Various attempts, though in different directions, have been made to study the growth of such functions.

Sato [59,60] adopted an elegant method to study the growth of

of 'index' of an entire function. Thus, if

$$(1.4.1) \qquad \rho(q) = \limsup_{r \to \infty} \frac{\log^{[q]} M(r)}{\log r}$$

and

$$(1.4.2) \qquad \kappa(q) = \limsup_{r \to \infty} \frac{\log^{[q-1]} M(r)}{r^{\rho(q)}} , \quad (0 < \rho(q) < \infty)$$

where $\log^{[0]} M(r) = M(r)$ and $\log^{[q]} M(r) = \log (\log^{[q-1]} M(r))$, $q=2,3,\ldots$ then $f(z)$ is said to be of index q if $\rho(q-1) = \infty$ and $\rho(q) < \infty$. The coefficient equivalents of $\rho(q)$ and $\kappa(q)$ for the entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of index q, as obtained by Sato, are as following:

$$(1.4.3) \qquad \rho(q) = \limsup_{n \to \infty} \frac{n \log^{[q-1]} n}{\log |a_n|^{-1}}$$

and

$$(1.4.4) \qquad \kappa(q) = \limsup_{n \to \infty} |a_n|^{\rho(q)/n} \log^{[q-2]} \frac{n}{e\rho(q)} \quad ; \ (0 < \rho(q) < \infty), q=2,3,\ldots$$

A formula analogous to (1.3.8) giving $\rho(q)$ for entire functions satisfying (1.3.6) has recently been obtained by Bajpai [6] .

For the entire function $f(z)$, Shah and Ishaq [71] considered

$$(1.4.5) \qquad \begin{matrix} \overline{\rho}(k) \\ \overline{\lambda}(k) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[k]} M(r)}{\log^{[k-1]} r} , \quad (0 \le \overline{\lambda}(k) \le \overline{\rho}(k) \le \infty)$$

and

$$(1.4.6) \qquad \begin{matrix} \overline{T}(k) \\ \overline{\tau}(k) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[k]} M(r)}{\log^{[k]} r} , \quad (1 \le \overline{\tau}(k) \le \overline{T}(k) \le \infty),$$

where k is some fixed integer not less than 2, and obtained results analogous to (1.3.2) and first part of (1.3.7). The upper limit in (1.4.6) for $k = 2$ is called logarithmic order which was first introduced by Iyer [29] to study the growth of entire functions of zero order. Further work in this direction has been done by Juneja [31] and Awasthi [4] .

Rahman [54] considered $r^{\rho}(\log r)^{\alpha_1} \ldots (\log^{[k]} r)^{\alpha_k}$ ($\log^{[0]} r = r$, $\log^{[k]} r = \log(\log^{[k-1]} r)$, $k = 1,2,3,\ldots$) as the comparison function to study the growth of $\log M(r)$. Here $\rho\,(0 < \rho < \infty)$ is the order of the function and $\{\alpha_i\}_{i=1}^{k}$ are nonnegative constants. He showed that if $T(\rho)$ and $\tau(\rho)$ denote the upper and lower limits of

$$r^{-\rho}(\log r)^{-\alpha_1} \ldots\ldots (\log^{[k]} r)^{-\alpha_k} \log M(r)$$

and Θ and θ those of

$$\rho^{\alpha_1} n(\log n)^{-\alpha_1} \ldots (\log^{[k]} n)^{-\alpha_k} |a_n|^{\rho/n} ,$$

then,

$$(1.4.7) \qquad e\,\rho\,T(\rho) = \Theta ,$$

$$(1.4.8) \qquad e\,\rho\,\tau(\rho) \geq \theta$$

and equality holds in (1.4.8) if (1.3.6) is satisfied. Further, if $f(z)$ is of logarithmic order $\alpha_1 (1 < \alpha_1 < \infty)$ and

$$\begin{matrix} u \\ v \end{matrix} = \lim_{n\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \left\{ \frac{(\alpha_1-1)^{\alpha_2}}{\alpha_1^{\alpha_1}} \frac{n^{\alpha_1}}{(\log n)^{\alpha_2} \ldots (\log^{[k-1]} n)^{\alpha_k}} \left(\frac{\alpha_1-1}{\log |a_n|^{-1}} \right)^{\alpha_1-1} \right\},$$

then,

(1.4.9) $\quad T(0) = u$,

(1.4.10) $\quad \tau(0) \geq v$

and equality holds in (1.4.10) if (1.3.6) is satisfied.

For a more precise specification of the rate of growth of $f(z)$, than given by (1.2.2) and (1.2.3), the concept of a proximate order has been used, which is more closely linked with $\log M(r)$. Thus the real valued function $\rho(r)$ is called a Lindelöf proximate order [16, p. 54] for an entire function $f(z)$ of order $\rho\,(0 < \rho < \infty)$, if it satisfies the following conditions:

(i) $\rho(r)$ is nonnegative, continuous and piecewise differentiable for $r > r_o$;

(ii) $\lim\limits_{r \to \infty} \rho(r) = \rho$;

(iii) $\lim\limits_{r \to \infty} r\rho'(r) \log r = 0$, where $\rho'(r)$ is either the right or the left derivative of $\rho(r)$ at points where they are different;

(iv) $\limsup\limits_{r \to \infty} \dfrac{\log M(r)}{r^{\rho(r)}} = 1$.

Valiron [89, p. 65] proved that there exists a proximate order for every entire function of finite positive order. His proof is based on the results of Blumenthal. Shah [64] has given a simple alternative proof of the existence of proximate orders without making use of the special properties of $M(r)$. Relations involving a proximate order and coefficients of the entire series have also been obtained [44, p. 42] .

Shah has defined and proved [65] the existence of a lower proximate order for every entire function of nonzero finite lower order λ.

Analogous to proximate order, Srivastava and Juneja [82] have introduced the concept of proximate type and showed that for every entire function of nonzero finite order and nonzero finite type there exists a proximate type. In a similar manner, the concepts of logarithmic proximate order, logarithmic proximate type etc. can be introduced. Further work in this direction has been done by Juneja [31], Awasthi [4] and G.S. Srivastava [76].

1.5. It is clear that formulae given in sections 1.3 and 1.4 are not, in general, applicable to an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if the coefficients a_n's vanish for infinitely many values of n. For studying the growth of such functions in terms of coefficients the entire series with gaps is considered. Thus, let

$$(1.5.1) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

be an entire function, where $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that no element of the sequence $\{a_n\}_{n=1}^{\infty}$ is zero*. It can be easily seen that results analogous to (1.3.2) and (1.3.3) hold for the entire series (1.5.1), if in those formulae, n is replaced by λ_n.

Juneja and Singh [38] obtained a coefficient formula for the lower order of the entire series (1.5.1). Thus, if f(z), given by (1.5.1),

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* Throughout this work, whenever we consider a power series of the form (1.5.1), it is assumed that $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that no element of the sequence $\{a_n\}_{n=1}^{\infty}$ is zero.

is of lower order $\lambda\,(0 \leq \lambda \leq \infty)$, then

$$(1.5.2) \qquad \lambda \geq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}}$$

and further, if

$$(1.5.3) \qquad \psi(n) \equiv |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)} \text{ is a nondecreasing function}$$

of n for $n > n_o$, then

$$(1.5.4) \qquad \lambda = \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} .$$

The formulae for the order and lower order of the entire series (1.5.1) involving the ratio of the two consecutive coefficients were obtained by Awasthi [4] who showed that if $f(z)$, defined by (1.5.1), is of order ρ and lower order $\lambda\,(0 \leq \lambda \leq \rho \leq \infty)$ and if $\lambda_n \sim \lambda_{n+1}$ as $n \to \infty$, then

$$(1.5.5) \quad \liminf_{n \to \infty} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log |a_n/a_{n+1}|} \leq \lambda \leq \rho \leq \limsup_{n \to \infty} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log |a_n/a_{n+1}|}$$

and further, if (1.5.3) holds, then

$$(1.5.6) \qquad \begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{n \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log |a_n/a_{n+1}|} .$$

Recently, Basinger [8] has obtained the lower type of $f(z)$ in terms of the coefficients of its Taylor series (1.5.1). He has shown that if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is an entire function of order $\rho\,(0 < \rho < \infty)$ and lower type $t\,(0 \leq t \leq \infty)$ such that $\liminf_{n \to \infty} \frac{\lambda_{n-1}}{\lambda_n} = L\ (> 0)$, then

$$(1.5.7) \qquad e\,\rho\,t \geq L \liminf_{n \to \infty} \lambda_n\, |a_n|^{\rho/\lambda_n}$$

and further, if (1.5.3) holds, then

(1.5.8) $$e \rho t \leq \liminf_{n \to \infty} \lambda_n |a_n|^{\rho/\lambda_n} .$$

Macintyre [47] showed that if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function such that

(1.5.9) $$\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$$

then $f(z)$ is unbounded on $\text{Re } z > 0$. By taking the growth of the function $f(z)$ into consideration Edrei [21] weakened the condition (1.5.9). Thus he showed that if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is an entire function of finite order ρ such that

(1.5.10) $$\liminf_{S \to \infty} \frac{1}{\log S} \sum_{\lambda_n \leq S} \lambda_n^{-1} < \frac{1}{2\rho} ,$$

then $f(z)$ is unbounded in $\text{Re } z > 0$. From both these theorems it follows that $f(z)$ has no finite radial asymptotic values. Both Macintyre and Edrei showed that their growth conditions are best possible. Recently Anderson and Binmore [3] have generalized and sharpened the above results.

1.6. Let $f(z)$ be an entire transcendental function defined by the power series (1.5.1). Since the series converges for all finite z, the modulii of its terms; viz.,

$$|a_0| , |a_1| r^{\lambda_1} ,\ldots, |a_n| r^{\lambda_n} , \ldots$$

must decrease after some value of n, for any finite r. Hence there is one term of the series whose absolute value is not less than that of any other term. The modulus of this term we denote by $\mu(r)$. Thus

$$\mu(r) \equiv \mu(r,f) = \max_{n \geq 0} \{|a_n| \, r^{\lambda_n}\} .$$

Let,

$$\nu(r) = \max \{\lambda_n : \mu(r) = |a_n| \, r^{\lambda_n}\} .$$

$\mu(r)$ is called the maximum term of $f(z)$ for $|z| = r$ and $\nu(r)$ is called the rank of the maximum term $\mu(r)$ or central index of $f(z)$ for $|z| = r$. The function $\nu(r)$ is a nondecreasing, unbounded function of r, which is constant in intervals and has only ordinary discontinuties. Elements in the range set of $\nu(r)$ are called principal indices.

The maximum term and the central index of $f(z)$ have played a significant role in the study of the growth of an entire function. Valiron [89 , pp. 28-32] , by constructing a Newton's polygon, has established the following relations involving the maximum term $\mu(r)$, the central index $\nu(r)$ and the maximum modulus $M(r)$ of an entire function:

(1.6.1) $$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(t)}{t} \, dt \quad (0 \leq r_0 < r < \infty)$$

(1.6.2) $$\mu(r) \leq M(r) < \mu(r) \{1 + 2\nu \, (r + \frac{r}{\nu(r)})\} .$$

In 1957, Erdős [23] conjectured that for an entire (transcendental) function if U and u are superior and inferior limits of $\mu(r)/M(r)$ respectively, then either $U > u$ or $U = 0$. Gray and Shah [25] showed the conjecture to be true except in the case when the power series of $f(z)$ has 'wide latent gaps'. Clunie and Hayman [20] showed that conjecture is not true in general and have given the construction of a certain class of entire functions for which it is valid.

Valiron [89, p. 32] has shown that for functions of finite order ρ

(1.6.3) $$\log M(r) \sim \log \mu(r) \text{ as } r \to \infty$$

and that

(1.6.4) $$\rho = \limsup_{r \to \infty} \frac{\log \log \mu(r)}{\log r} = \limsup_{r \to \infty} \frac{\log \nu(r)}{\log r}.$$

Whittaker [95] obtained an analogous result for the lower order λ. Thus, he proved that if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is an entire function of order ρ and lower order λ, then

(1.6.5) $$\lambda = \liminf_{r \to \infty} \frac{\log \log \mu(r)}{\log r} = \liminf_{r \to \infty} \frac{\log \nu(r)}{\log r},$$

and

(1.6.6) $$\lambda \leq \rho \liminf_{n \to \infty} \frac{\log \lambda_n}{\log \lambda_{n+1}}.$$

From (1.6.6) it follows that if an entire function is of regular growth, then,

$$\lim_{n \to \infty} \frac{\log \lambda_n}{\log \lambda_{n+1}} = 1.$$

Shah and Ishaq [71] obtained results analogous to (1.6.4) for the constants defined by (1.4.5) and (1.4.6).

Shah, in 1942, improved the following result of Pólya and Szegö [52]

(1.6.7) $$\liminf_{r \to \infty} \frac{\nu(r)}{\log \mu(r)} \leq \rho \leq \limsup_{r \to \infty} \frac{\nu(r)}{\log \mu(r)}$$

and showed that [61]

(1.6.8) $$\liminf_{r \to \infty} \frac{\nu(r)}{\log \mu(r)} \leq \lambda \leq \rho \leq \limsup_{r \to \infty} \frac{\nu(r)}{\log \mu(r)}.$$

Growth numbers γ and δ of an entire function $f(z)$, of order $\rho(0 < \rho < \infty)$, are defined by

$$(1.6.9) \qquad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\nu(r)}{r^{\rho}} = \begin{matrix} \gamma \\ \delta \end{matrix} .$$

Shah [66] proved that

$$(1.6.10) \qquad \delta \leq (\gamma/e)\, e^{\delta/\gamma} \leq \rho\, T \leq \gamma$$

and

$$(1.6.11) \qquad \delta \leq \rho\, t \leq \delta(1 + \log(\gamma/\delta)) \leq \gamma$$

where T and t are the type and the lower type of $f(z)$. He also noted [66] that

$$(1.6.12) \qquad \delta/\gamma\rho \leq c \leq d \leq \gamma/\delta\rho$$

where

$$\begin{matrix} c \\ d \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r)}{\nu(r)} .$$

S.K. Singh [74] proved that one cannot have simultaneously

$$(1.6.13) \qquad \gamma + \delta = e\, \rho\, T \text{ and } \delta = \rho\, T.$$

For $0 \leq \rho \leq \infty$, Shah [67] proved that

$$(1.6.14) \qquad \limsup_{r \to \infty} \frac{\log \mu(r)}{\nu(r) \log r} \leq 1 ,$$

and if for a sequence of values of r tending to infinity

$$\log\log \mu(r) = \{1 + o(1)\} \log\log r,$$

then,

$$(1.6.15) \qquad \limsup_{r \to \infty} \frac{\log \mu(r)}{\nu(r) \log r} = 1.$$

(1.6.14) was further sharpened by R.P. Srivastav [78] who showed that for $0 < \lambda, \ \rho < \infty$,

$$(1.6.16) \qquad \limsup_{r \to \infty} \frac{\log \mu(r)}{\nu(r) \log r} \leq 1 - \lambda/\rho$$

and

$$(1.6.17) \qquad \limsup_{r \to \infty} \frac{\log \mu(r)}{\nu(r) \log \nu(r)} \leq \frac{1}{\lambda} - \frac{1}{\rho} \ .$$

An asymptotic relation between $\nu(r)$ and $\log \mu(r)$ was also obtained by Shah [70] who showed that $\nu(r) \sim \rho T r^{\rho}$ as $r \to \infty$ if and only if $\log \mu(r) \sim T r^{\rho}$.

Further results involving $M(r)$, $\mu(r)$, $\nu(r)$, order, type etc. of the function $f(z)$ have been obtained by R.S.L. Srivastava [80] , Rahman [55] , Clunie [17] , R.S.L. Srivastava and O.P. Juneja [83] , G.S. Srivastava [77] and others.

1.7. Let

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

be the Nevanlinna's characteristic of the entire function $f(z)$, where $\log^+ x = \max(\log x, 0)$. It is known [27, p. 8] that $m(r,f)$ is an increasing convex function of $\log r$ and has an integral representation similar to (1.2.1). The growth of the function $f(z)$ can be studied in terms of its Nevanlinna's characteristic also. In fact, in view of the fundamental inequality (see e.g. [27, p. 18])

$$(1.7.1)\quad m(r,f) \leq \log M(r,f) \leq \frac{R+r}{R-r}\, m(R,f), \quad (0 \leq r < R),$$

it follows that the order ρ and lower order λ of the entire function $f(z)$ are given by

$$(1.7.2)\qquad \begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log m(r,f)}{\log r}.$$

Since both $\log M(r,f)$ and $m(r,f)$ are nonnegative, increasing, convex functions of $\log r$, a natural question that arises is that if $W(r)$ is a nonnegative, increasing convex function of $\log r$, does there exist an entire function $f(z)$ which satisfies either (i) $\log M(r,f) = W(r)$ or (ii) $m(r,f) = W(r)$? The answer to this question is not, in general, in the affirmative [45].

Valiron [88] showed that if $W(r)$ is a function given by

$$(1.7.3)\qquad W(r) = \text{const.} + \int_{\alpha}^{r} \frac{\Lambda(t)}{t}\, dt \quad (r \geq \alpha > 0)$$

where $\Lambda(t)$ is nonnegative, nondecreasing and unbounded then there exists an entire function of finite order such that

$$(1.7.4) \qquad \log M(r,f) \sim W(r), \quad \text{as} \quad r \to \infty,$$

provided

$$(1.7.5) \qquad W(r) < r^K$$

for some $K > 0$ and for sufficiently large values of r.

A similar problem for the function $m(r,f)$ was treated by A. Edrei and W.H.J. Fuchs [22]. They proved that if $W(r)$ is defined as in (1.7.3) and if (1.7.5) is satisfied then there exists an entire function $f(z)$ of finite order such that

$$(1.7.6) \qquad \log f(r) = \log M(r,f) \sim m(r,f) \sim W(r), \text{ as } r \to \infty,$$

Clunie [19] generalized the above results when he showed that if $W(r) \neq O(\log r)$ as $r \to \infty$, then there exists an entire function $f(z)$ satisfying (1.7.6).

Another interesting situation is the existence of entire transcendental functions of arbitrarily fast or arbitrarily slow growth. It is well known ([13], [42], [51], [91]) that there exist entire transcendental functions whose rate of growth as measured by their maximum modulii is arbitrarily fast or arbitrarily slow. B. Lepson [43] has recently shown that given

two positive functions $h(r)$ and $k(r)$ for $r > 0$ such that $\log k(r) \neq O(\log r)$, there exists an entire function $f(z)$ and two sequences $\{c_n\}$ and $\{r_n\}$ of positive numbers tending to infinity such that for every positive integer n, $M(c_n, f) > h(c_n)$ and $M(r_n, f) < k(r_n)$.

D. Gair [24] showed that there exists an entire function of order ρ and type T such that

$$(1.7.7) \qquad \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^\rho} = T$$

for all θ, $0 \leq \theta \leq 2\pi$. Recently J.M. Anderson [2] further showed that for an entire function $f(z)$ of order $\rho > 0$ and type T, the set of values of θ for which limit exists in (1.7.7) is of first Baire category in $[0, 2\pi)$.

1.8. A set of real valued continuous functions $\Phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$ on a closed interval $[a,b]$ is called a Chebyshev system if each polynomial $P = a_1 \phi_1 + \ldots + a_n \phi_n$, having real coefficients $\{a_i\}_{i=1}^{n}$ such that at least one a_i is nonzero, has atmost $n-1$ distinct zeros on $[a,b]$.

Let f be continuous real valued function on $[a,b]$ with its norm defined as

$$||f|| = \sup_{a \leq x \leq b} |f(x)|$$

and let

$$(1.8.1) \qquad E_n^{\Phi}(f) = \inf_{a_1, \ldots, a_n} ||f - (a_1\phi_1 + \ldots + a_n\phi_n)||$$

denote n^{th} degree of approximation of f by Chebyshev system Φ. If the infimum in (1.8.1) is attained for a linear combination $P = b_1 \phi_1 + \dots + b_n \phi_n$ then P is called a linear combination of best approximation.

It is well known [46, pp. 26-27] that for a Chebyshev system there is a unique polynomial of best approximation for each continuous function and conversely if $\phi_1, \phi_2, \dots, \phi_n$ are continuous functions defined on $[a,b]$ and if each continuous f has only one polynomial of best approximation then $\{\phi_1, \phi_2, \dots, \phi_n\}$ is a Chebyshev system.

Let

$$C_n(x) = \tfrac{1}{2} \{ (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \} \quad , n = 0,1,2,\dots.$$

be the Chebyshev polynomials. Then it follows that $P_n(x) = 2^{-n+1} C_n(x)$ is the polynomial of best approximation for the continuous function x^n on $[-1,1]$ with respect to the Chebyshev system $w = \{1, x, \dots, x^{n-1}\}$.

A real or complex-valued function f defined on $I \equiv [-1,1]$ is called analytic on I if there exists an analytic extension of f onto some open set G of the complex plane that contains I. The degree of approximation, $E_n(f)$, of such functions f, by algebraic polynomials, has many interesting properties. Bernstein [9, p. 117] showed that a function f, defined on $[-1,1]$ is analytic in this interval, if and only if, $\limsup_{n \to \infty} E_n^{1/n}(f) < 1$. In particular, $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$, if and only if,

$$\lim_{n \to \infty} E_n^{1/n}(f) = 0.$$

He further showed that the rate of decrease of $E_n^{1/n}(f)$ depends on the order and type of the entire function $f(z)$. Thus, there exists a constant $\rho > 0$ such that

$$\limsup_{n \to \infty} n^{1/\rho} E_n^{1/n}(f) \tag{1.8.2}$$

is finite if and only if $f(x)$ is the restriction to $[-1,1]$ of an entire function of order ρ and some finite type T. Varga [92] strengthened it further when he showed that

$$\limsup_{n \to \infty} \frac{n \log n}{\log[1/E_n(f)]} = \rho \tag{1.8.3}$$

where ρ is a nonnegative real number if and only if $f(x)$ is the restriction to $[-1,1]$ of an entire function of order ρ.

The above results of Bernstein and Varga have been extended by A.R. Reddy [56] to entire functions of index k as defined by Sato, and to entire functions of logarithmic order $\bar{\rho}(1 < \bar{\rho} < \infty)$. He has also obtained the corresponding analogues for the lower order and lower type for a subclass of entire functions.

Recently J.P. Singh [73] has obtained a characterization for the lower order of an entire function $f(z)$ in terms of the degree of approximation $E_n(f)$, of its restriction to $[-1,1]$. Thus, he has shown that if $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$ of lower order $\lambda(0 < \lambda < \infty)$, then

$$\operatorname*{Min}_{\{n_h\}} \limsup_{h \to \infty} \left\{ \frac{\log \left(\frac{1}{E_{n_h}(f)} \right)}{n_h \log n_{h-1}} \right\} = \frac{1}{\lambda} \tag{1.8.4}$$

where minimum in (1.8.4) is taken over all increasing sequences $\{n_h\}$ of natural numbers.

The results of Bernstein have been extended in a different direction also. Thus, results have been obtained about the degree of approximation of an analytic function on rather general sets of the complex plane (see e.g. [93, Chapters III & IV]). Let $p_n(z)$ be the best approximation to $f(z)$ on a set D of the complex plane. Set,

$$E_n = ||f(z) - p_n(z)||_D = \max_{z \in D} |f(z) - p_n(z)|,$$

then for $D = \{z: |z| \leq r\}$, a direct application of techniques from the theory of entire functions gives that $f(z)$ is entire of order ρ and type T if and only if

$$\lim_{n \to \infty} n^{1/\rho} E_n^{1/n} = \frac{1}{r} (e\rho T)^{1/\rho} . \tag{1.8.5}$$

Recently J.R. Rice [57] has extended the above result to more general sets. Thus, let C be the point set in the complex plane whose complement is connected and regular, $f(z)$ be defined on C and $P_n^*(z)$ be the best polynomial approximation to $f(z)$ on C. Then, Rice proved that

$$\lim_{n \to \infty} n^{1/\rho} ||f(z) - P_n^*(z)||_C^{1/n} = d_\infty(C) (e\rho T)^{1/\rho} \tag{1.8.6}$$

if and only if $f(z)$ is an entire function of order $\rho > 0$ and type $T(0 < T < \infty)$, $d_\infty(C)$ being the transfinite diameter of C.

1.9. Let U denote the class of functions which are analytic in the unit disc $D = \{z: |z| < 1\}$. Analogous to the case of entire functions the growth of this class of functions has been studied in terms of their maximum modulus $M(r) \equiv M(r,f) = \max_{|z|=r} |f(z)|$, $(0 < r < 1)$. Thus, the order ρ_o and lower order λ_o of a function $f(z) \varepsilon U$ are defined as

$$(1.9.1) \qquad \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^+ \log^+ M(r)}{-\log(1-r)} = \begin{matrix} \rho_o \\ \lambda_o \end{matrix},$$

where $\log^+ x = \max(\log x, 0)$. As usual a function $f(z) \varepsilon U$ is said to be of regular growth if $\rho_o = \lambda_o$ and is of irregular growth if $\rho_o > \lambda_o$.

It can be easily seen that the order ρ_o and lower order λ_o of $f(z) \varepsilon U$, defined by (1.9.1), are, in general, different from the Nevanlinna order ρ_N and lower Nevanlinna order λ_N defined as

$$\begin{matrix} \rho_N \\ \lambda_N \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^+ m(r)}{-\log(1-r)},$$

where,

$$m(r) \equiv m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta, \qquad (0 < r < 1).$$

In fact in view of the fundamental inequality (1.7.1) (which is true for functions of the class U if $\log M(r)$ is replaced by $\log^+ M(r)$), it follows by taking $r = 2R - 1$ that

$$\rho_N \leq \rho_o \leq \rho_N + 1$$

and

$$\lambda_N \leq \lambda_o \leq \lambda_N + 1.$$

Concepts of the maximum term and central index can also be introduced for a function analytic in the unit disc, in the same manner as done for entire functions. Thus, if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is a function of class $\mathcal{U}$, then its maximum term $\mu(r)$ and central index $\nu(r)$ are defined for $0 < r < 1$, by

$$\mu(r) \equiv \mu(r,f) = \max_{n \geq 0} \{|a_n| r^{\lambda_n}\} \text{ and } \nu(r) = \max \{\lambda_n : \mu(r) = |a_n| r^{\lambda_n}\}.$$

Valiron [90] has shown that for a function of class $\mathcal{U}$ having positive order

$$\log \mu(r) = \log \mu(r_o) + \int_{r_o}^{r} \frac{\nu(t)}{t} dt, \tag{1.9.2}$$

and

$$\mu(r) \leq M(r) < \mu(r) \{1+2\nu(r + \frac{1-r}{\nu(r)})\} \frac{1}{1-r}, \quad (0 < r_o < r < 1). \tag{1.9.3}$$

L.R. Sons [75] obtained the order and lower order of a function $f(z) \in \mathcal{U}$ in terms of its maximum term and the central index. Thus, she showed that if $f(z) \in \mathcal{U}$ has order ρ_o $(0 < \rho_o < \infty)$ and lower order λ_o $(0 \leq \lambda_o < \infty)$, then

$$\begin{matrix}\rho_o \\ \lambda_o\end{matrix} = \lim_{r \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log \mu(r)}{-\log (1-r)} \tag{1.9.4}$$

and

$$\begin{matrix}1+\rho_o \\ 1+\lambda_o\end{matrix} = \lim_{r \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \nu(r)}{-\log(1-r)}. \tag{1.9.5}$$

Further, if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, then

$$(1.9.6) \qquad 1 + \lambda_0 \leq (1 + \rho_0) \liminf_{n \to \infty} \frac{\log \lambda_n}{\log \lambda_{n+1}} .$$

Analogous to the case of entire functions, (1.9.6) shows that if $f(z) \in U$ is of regular growth then

$$\lim_{n \to \infty} \frac{\log \lambda_n}{\log \lambda_{n+1}} = 1.$$

Beuermann [10] and MacLane [49, p. 47] separately obtained the order ρ_0 of $f(z) \in U$ in terms of the coefficients of its power series $\sum_{n=0}^{\infty} a_n z^n$. Their result is

$$(1.9.7) \qquad \frac{\rho_0}{1+\rho_0} = \limsup_{n \to \infty} \frac{\log^+ \log^+ |a_n|}{\log n} .$$

A function $f(z) \in U$ is said to have the asymptotic value a at ζ, $|\zeta| = 1$ if and only if there is an arc Γ in D tending to ζ such that $f(z) \to a$ on Γ. The asymptotic values of the function $f(z)$ are intimately connected with its growth. An attempt in this direction was made by G.R. Maclane who introduced the class A. The MacLane class A consists of those nonconstant functions in U for which there exists an everywhere dense set of points ζ on the unit circle with the following property: there exists an arc Γ in the disc D and a complex number a(possibly ∞) such that $f(z) \to a$ as $z \to \zeta$ on Γ.

MacLane [49] obtained the following sufficient conditions for a nonconstant function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ of the class U to belong to the class A :

$$(1.9.8) \qquad \liminf_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} > q > 3$$

$$(1.9.9) \qquad \int_0^1 (1-r) \log^+ M(r) < \infty$$

$$(1.9.10) \qquad \log^+ |a_n| < n^\lambda \text{ for some } \lambda \text{ such that } 0 < \lambda < 2/3 .$$

The condition (1.9.10) shows that all functions of the class $\mathcal{U}$ whose order is less than 2 belong to A .

The condition (1.9.9) was weakened by R. Hornblower [28] who proved that the condition

$$(1.9.11) \qquad \int_0^1 \log^+ \log^+ M(r) < \infty$$

is sufficient for a nonconstant function $f(z)$ of the class $\mathcal{U}$ to belong to the class A .

The functions of the class $\mathcal{U}$ satisfying (1.9.8) or (1.9.11) have another interesting property. Anderson [1] showed that if either (1.9.8) or (1.9.11) is satisfied for the function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ and if

(a) $\frac{\lambda_{n+1}}{\lambda_n} > q > 1 \quad (n = 0,1,2,\ldots)$

(b) $a_n \not\to 0$ as $n \to \infty$

then $f(z)$ assumes every finite value infinitely often in every neighbourhood of ζ , $|\zeta| = 1$.

A is not a linear space. Barth and Ryan [7] showed that the sum and product of two functions in A need not be in A. Brannan and Hornblower [15] not only gave an elementary proof of this fact but also showed that any nonconstant function of the class $\mathcal{U}$ may be represented both as a sum and a product of the pairs of the functions in A .

Recently, Pratt [53] has considered the class R of those functions of the class U which have radial limits zero at a dense subset of the unit circle $|z| = 1$ and showed that if $f(z) \varepsilon U$ then it can be expressed as a sum and product of functions of the class R.

1.10. The theory of functions analytic in the unit disc has been vastly enriched by contributions in the directions of univalent and multivalent functions, H^p spaces, theory of cluster sets, overconvergence etc. Similarly the theory of entire functions has been benefited by advances made by studying their exceptional values, Julia lines, asymptotic values, distribution of zeros etc. A good deal of literature can also be found dealing with entire functions of exponential type. Here we have briefly described only those topics in the theory in the direction of which we have tried to pursue the investigations further in the present work. The results are contained in the next seven chapters.

In Chapter two we make an attempt to classify entire functions according to their mode of growth. Our approach unifies the various approaches made earlier and is applicable to functions of both slow and fast growth. For this purpose we define (p,q)-order and lower (p,q)-order of an entire function and mainly obtain their coefficient characterizations. Results given here generalize and improve various known results in this direction.

In Chapter three we continue to study our classification scheme for the class of entire functions. We define (p,q)-type and lower (p,q)-type of an entire function and obtain their coefficient characterizations. Some results involving (p,q) type, lower (p,q)-type and ratio of two consecutive coefficients of the entire Taylor series are also obtained.

Further we define (p,q)-growth numbers of an entire function and derive formulae which determine them in terms of ratio of two consecutive coefficients of the entire Taylor series.

In fourth Chapter the results of second and third Chapters are used to determine the rate of decrease of the degree of Chebyshev approximation of a continuous function which is restriction to $[-1,1]$ of an entire function.

Fifth Chapter deals with the polynomial expansion of an entire function. Using results of second and third Chapters various results relating (p,q)-order, (p,q)-type etc. to the coefficient polynomials are obtained. Some of these results generalize results of Rice [57].

Next three chapters deal chiefly with the growth of functions of class $\mathcal{U}$ as measured by its maximum modulus, $\mathcal{U}$ being the class of nonconstant functions analytic in the unit disc.

In Chapter six we give complete coefficient characterizations for the lower order of the functions of class $\mathcal{U}$. We also give a sufficient condition for a function in $\mathcal{U}$ to belong to the MacLane class $\mathcal{A}$ consisting of those functions of the class $\mathcal{U}$ which have asymptotic values at a dense subset of the unit circle. This improves a similar result of MacLane [49]. Finally we show that there exist functions analytic in the disc $D_R = \{z: |z| < R\}$ whose upper rate of growth is arbitrarily fast and simultaneously whose lower rate of growth is arbitrarily slow.

In seventh Chapter we define type and lower type for functions of the class $\mathcal{U}$. Various formulae have been obtained which connect type and lower type of these functions with the coefficients of the corresponding

Taylor series. Growth numbers for this class of functions are defined and formulae determining them in terms of ratio of consecutive coefficients of the Taylor series are obtained.

In the last chapter we define a D-proximate order and obtain results relating D-proximate order and a nondecreasing function $\phi(r)$ which tends to infinity as $r \to 1$. The results when applied to the theory of analytic functions give relationships that exist between a given D-proximate order for a function analytic in the unit disc and its distribution of zeros, maximum term, maximum real part and geometric mean values etc.

CHAPTER 2

(p,q)-ORDER AND LOWER (p,q)-ORDER OF AN ENTIRE FUNCTION

2.1. Let

$$(2.1.1) \qquad f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$$

be a nonconstant entire function, where $\lambda_o = 0$ and $\{\lambda_k\}_{k=1}^{\infty}$ is the strictly increasing sequence of positive integers such that no element of the sequence $\{a_k\}_{k=1}^{\infty}$ is zero.

To estimate the growth of $f(z)$ precisely, the concept of order, as defined by (1.2.2), is used. The concept of type, as given by (1.2.3), is introduced to determine the relative growth of two entire functions of the same nonzero finite order. If $f(z)$ is of infinite or zero order, the definition of 'type' is not feasible and so growth of such functions cannot be precisely measured by confining to these concepts only. For entire functions of infinite order, Sato introduced the concept of 'index' of an entire function, as defined in Section 1.4 . However, there are two shortcomings in the classification introduced by Sato. First, it does not compare the growth of entire functions of zero order and secondly it does not give any precise information about the growth of those functions for which $\rho(q-1) = \infty$ and $\rho(q) = 0$. On the other hand, though the results of Shah and Ishaq as mentioned in section 1.4 give commendable results about functions of slow growth, they give a little information about the rate of growth of rapidly increasing functions. Moreover, since the concept of 'type' for entire functions with same $\bar{\rho}(k)$ or $\bar{T}(k)$ (as defined

in (1.4.5) and (1.4.6)) is not introduced, the comparison of growth of such functions is not possible.

In the present chapter we intend to classify entire functions further according to their mode of growth and make an attempt to find more precise information about the growth of an entire function, than given by the above concepts. We do not claim that our classification scheme assigns to every entire function a distinct growth constant (In fact such a classification is impossible, see e.g. [44, p. 383]). However, our approach does yield results which are applicable in a wider perspective and at the same time give refinements of the results obtained by various approaches made in this direction. It will be seen that our approach unifies the above mentioned approaches and is applicable to every entire function whether of slow growth or of fast growth. We introduce some new growth constants for this purpose and obtain their coefficient characterizations. The results which we obtain generalize, improve and sharpen many of the known results (see e.g. [34] , [60], [71] , [95] etc.).

2.2. For the entire function $f(z)$ which is not a polynomial of degree 1, set,

$$(2.2.1) \qquad \rho(p,q) = \limsup_{r \to \infty} \frac{\log^{[p]} M(r)}{\log^{[q]} r}$$

where p and q are integers such that $p \geq q \geq 0$, $M(r)$ is the maximum modulus of $f(z)$ for $|z| = r$, $\log^{[0]} x = x$ and $\log^{[p]} x = \log(\log^{[p-1]} x)$, $p = 1,2,\ldots$. It can be easily seen that if $p > q$ then $0 \leq \rho(p,q) \leq \infty$

and if $p = q$ then $1 \leq \rho(p,q) \leq \infty$. Moreover if $0 < \rho(p,q) < \infty$, then $\rho(p',q) = \infty$ for $p' < p$, $\rho(p,q') = 0$ for $q' < q$ and $\rho(p+n, q+n) = 1$ for $n = 1,2,\ldots.$ Further, it also follows that if $0 < \rho(p,q) < 1$, then for all pairs of integers (ζ_1, η_1) satisfying

$$\text{(i)} \qquad \eta = \zeta + q - p, \zeta < p$$

we have $\rho(\zeta_1, \eta_1) = 0$. Similarly if $1 < \rho(p,q) < \infty$, then for all pairs of integers (ζ_2, η_2) satisfying (i) we have $\rho(\zeta_2, \eta_2) = \infty$. Consequently, if $0 < \rho(p,q) < \infty$, $\rho(\zeta, \eta) = \infty$ for $\eta - \zeta > q - p$ and $\rho(\zeta, \eta) = 0$ for $\eta - \zeta < q - p$.

Thus we are led to the following definitions.

DEFINITION 2.1. *An entire function* $f(z)$ *is said to be of index pair* (p,q) $p \geq q \geq 1$ *if* $0 < \rho(p,q) < \infty$ *and* $\rho(p-1,q-1)$ *is not a nonzero finite number. If* $\rho(p,q)$ *is never nonzero finite and* $\rho(p,1) = 0$ *for some* p, *then index pair of* $f(z)$ *is defined as* $(m,1)$ *where* $m = \inf\{p : \rho(p,1) = 0\}$. *If* $\rho(p,q)$ *is always infinite, index pair of* $f(z)$ *is defined to be* (∞, ∞).

DEFINITION 2.2. *An entire function* $f(z)$ *is said to be of* (p,q)*-order* ρ *if* $f(z)$ *is of index pair* (p,q), $p \geq q \geq 1$ *and* $b < \rho(p,q) \equiv \rho < \infty$, *where* $b = 0$ *or* 1 *according as* $p > q$ *or* $p = q$.

We note that entire functions of index pair $(1,1)$ are polynomials of degree greater than 1, those with index pair $(2,1)$ are transcendental entire functions of finite order and entire functions having index pair $(p,1)$ belong to the class of functions studied by Sato [59]. Similarly, it may be noted that entire functions having index pair (k,k) or $(k,k-1)$ are those studied by Shah and Ishaq [71].

Let $f_1(z)$ be an entire function of (p,q)-order ρ_1 and $f_2(z)$ be an entire function of (p',q')-order ρ_2 and let $p \leq p'$. Following results about their comparative growth can be easily deduced:

(a) If $p' - p > q' - q$, then the growth of $f_1(z)$ is slower than the growth of $f_2(z)$.

(b) If $p' - p < q' - q$, then the growth of $f_1(z)$ is faster than the growth of $f_2(z)$

(c) If $p' - p = q' - q > 0$, then growth of $f_1(z)$ is slower than growth of $f_2(z)$ if $\rho_2(p',q') \geq 1$ while growth of $f_1(z)$ is faster than the growth of $f_2(z)$ if $\rho_2(p',q') < 1$.

(d) Let $p'-p = q'-q = 0$. Then $f_1(z)$ and $f_2(z)$ are of same index pair (p,q). If $\rho_1(p,q) > \rho_2(p,q)$ then growth of $f_1(z)$ is faster than growth of $f_2(z)$ and if $\rho_1(p,q) < \rho_2(p,q)$ then growth of $f_1(z)$ is slower than growth of $f_2(z)$. If $\rho_1(p,q) = \rho_2(p,q)$ then (2.2.1) does not give any precise estimate about the relative growth of $f_1(z)$ and $f_2(z)$. For such functions we shall introduce the concept of '(p,q)-type' in the next chapter.

We define the lower (p,q)-order for the entire function $f(z)$ as follows:

DEFINITION 2.3. An entire function $f(z)$ of index pair (p,q) is said to be of lower (p,q)-order λ if

$$(2.2.2) \qquad \lambda \equiv \lambda(p,q) = \liminf_{r \to \infty} \frac{\log^{[p]} M(r)}{\log^{[q]} r}, \qquad (0 \leq \lambda < \infty).$$

An entire function for which (p,q)-order and lower (p,q)-order are equal is said to be of regular (p,q)-growth. Functions which are not of regular (p,q)-growth are said to be of irregular (p,q)-growth.

To avoid certain trivial cases, throughout our discussions we take $p \geq 2$.

The following notations are frequently used throughout this work.

NOTATION 1. $\exp^{[0]} x = \log^{[0]} x = x$; $\exp^{[m]} x = \log^{[-m]} x = \exp(\exp^{[m-1]} x) = \log(\log^{[-m-1]} x)$, $m = 0, \pm 1, \ldots$. We wish to point out here that in our calculations wherever $(\log^{[m]} x)^{\alpha}$ $(0 \leq \alpha < \infty)$ occurs, it is understood that x is such that this expression is a real number.

NOTATION 2.

$$E_{[r]}(x) = \prod_{j=0}^{r} \exp^{[j]} x \quad ; \quad \Lambda_{[r]}(x) = \prod_{j=0}^{r} \log^{[j]} x$$

$$E_{[-r]}(x) = \frac{x}{\Lambda_{[r-1]}(x)} \quad ; \quad \Lambda_{[-r]}(x) = \frac{x}{E_{[r-1]}(x)} ,$$

$r = 0, \pm 1, \ldots\ldots$

NOTATION 3.

$$\begin{aligned} P_{\tau}(\alpha) \equiv P_{\tau}(\alpha,p,q) &= \alpha \quad \text{if } p > q \\ &= \tau + \alpha \text{ if } p = q = 2 \\ &= \max(1,\alpha) \text{ if } p = q \geq 3, \end{aligned}$$

where $0 \leq \alpha \leq \infty$ and $0 \leq \tau \leq 1$. We shall write $P(\alpha)$ for $P_1(\alpha)$.

2.3. In this section we find some results which connect (p,q)-order and lower (p,q)-order of the entire function $f(z)$ with its maximum term $\mu(r)$ and the central index $\nu(r)$ as defined in section 1.6 . The results are given in the form of lemmas.

Let,

$$(2.3.1) \qquad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \nu(r)}{\log^{[q]} r} = \begin{matrix} \Theta \\ \theta \end{matrix}$$

and

$$(2.3.2) \qquad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \mu(r)}{\log^{[q]} r} = \begin{matrix} \Phi \\ \phi \end{matrix}$$

then we have the following:

LEMMA 2.1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function having* (p,q)-*order* ρ *and let* Θ *and* Φ *be given by* (2.3.1) *and* (2.3.2) *respectively, then*

$$(2.3.3) \qquad \rho = P(\Theta) = \Phi .$$

PROOF. In view of (1.6.1) it follows easily that $\Phi = P(\Theta)$. We now prove $\rho = P(\Theta)$. First let $\Theta < \infty$. Then by (2.3.1), for every $\varepsilon > 0$ and for all $r > r_o = r_o(\varepsilon)$,

$$\nu(r) < \exp^{[p-2]} (\log^{[q-1]} r)^{\Theta+\varepsilon}$$

and in view of the relation $\Phi = P(\Theta)$, we get

$$\log \mu(r) < \exp^{[p-2]} (\log^{[q-1]} r)^{P(\Theta)+\varepsilon}$$

for all $r > r_1 = r_1(\varepsilon)$. Now, by (1.6.2), for all $r > \max(r_o, r_1)$, we get

$$\log M(r) < \log \mu(r) + \log \nu\left(r + \frac{r}{\nu(r)}\right) + o(1)$$

$$< \exp^{[p-2]} (\log^{[q-1]} r)^{P(\Theta)+\varepsilon} + \exp^{[p-3]} (\log^{[q-1]} 2r)^{\Theta+\varepsilon} + o(1)$$

which implies

$$\frac{\log^{[p]} M(r)}{\log^{[q]} r} < P(\Theta) + \varepsilon ,$$

and so proceeding to limit, we get $\rho \leq P(\Theta)$. This inequality is obviously true if $P(\Theta) = \infty$. Next, in view of the inequality $\mu(r) \leq M(r)$ and the relation $\Phi = P(\Theta)$, it follows that $\rho \geq P(\Theta)$ and consequently (2.3.3) follows.

REMARK. For $(p,q) = (2,1)$ this result is due to Valiron [89, p. 33] and for $(p,q) = (p,p)$ or $(p,p-1)$ it is due to Shah and Ishaq [71] .

LEMMA 2.2. *Let*

(i) $\phi(x)$ *be a positive increasing function of* x *for* $x > 0$

(ii) $\liminf\limits_{x \to \infty} \dfrac{\log^{[p-1]} \phi(x)}{\log^{[q]} x} = \alpha \; (0 \leq \alpha < \infty)$.

Then corresponding to each pair of positive numbers β , γ *satisfying the inequalities*

$$\alpha < \beta, \quad \alpha/\beta < \gamma < 1$$

there is a sequence $\{x_n\}$ *of positive numbers tending to infinity such that*

$$\log^{[p-1]} \phi(x) < \beta \log^{[q]} x \quad \text{for} \quad \exp^{[q-1]} (\log^{[q-1]} x_n)^{\gamma} \leq x \leq x_n.$$

PROOF. Let $\{x_n\}$ be a sequence of positive numbers such that

$$\frac{\log^{[p-1]} \phi(x_n)}{\log^{[q]} x_n} < \beta \gamma.$$

Then, if $\exp^{[q-1]} (\log^{[q-1]} x_n)^{\gamma} \leq x \leq x_n$, we get

$$\log^{[p-1]} \phi(x) \leq \log^{[p-1]} \phi(x_n) < \beta \gamma \log^{[q]} x_n \leq \beta \log^{[q]} x.$$

Hence the lemma.

REMARK. For $(p,q) = (2,1)$ this lemma was proved by Whittaker [95] .

LEMMA 2.3. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having lower (p,q)-order λ , then

(2.3.4) $$\lambda = P(\theta) = \phi$$

where θ and ϕ are as defined in (2.3.1) and (2.3.2) respectively.

PROOF. Using (1.6.1) it follows easily that $\phi = P(\theta)$. Since $\mu(r) \leq M(r)$, we get $P(\theta) \leq \lambda$. Thus we need only prove $P(\theta) \geq \lambda$. Let $\theta < \infty$. Since $\nu(r)$ is an increasing function of r, by Lemma 2.2 for $\theta < \beta$, $\theta/\beta < \gamma < 1$, there exists a sequence $\{R_n\}$ of positive numbers tending to infinity such that

(2.3.5) $$\frac{\log^{[p-1]} \nu(r)}{\log^{[q]} r} < \beta \text{ for } \exp^{[q-1]}(\log^{[q-1]} R_n)^{\gamma} \leq r \leq R_n.$$

Let δ and ε be positive numbers such that $\gamma < \delta < 1$, $\gamma/\delta < \varepsilon < 1$ and write $S_n = \exp^{[q-1]} (\log^{[q-1]} R_n)^{\delta}$ so that for sufficiently large n,

$$\exp^{[q-1]} (\log^{[q-1]} R_n)^\gamma < \exp^{[q-1]} (\log^{[q-1]} S_n)^\varepsilon < S_n < \frac{1}{2} R_n.$$

By (1.6.1),

$$\log \mu(S_n) = \log \mu(S_n^\varepsilon) + \int_{S_n^\varepsilon}^{S_n} \frac{\nu(x)}{x}\, dx$$

and

$$\log \mu(S_n^\varepsilon) < \varepsilon\, \nu(S_n^\varepsilon) \log S_n$$

so that

$$\log \mu(S_n) \geq \log \mu(S_n^\varepsilon) + \nu(S_n^\varepsilon) \int_{S_n^\varepsilon}^{S_n} \frac{dx}{x}$$

$$> \log \mu(S_n^\varepsilon) + \frac{1-\varepsilon}{\varepsilon} \log \mu(S_n^\varepsilon) = \frac{1}{\varepsilon} \log \mu(S_n^\varepsilon) .$$

Thus, using (2.3.5)

$$(1-\varepsilon) \log \mu(S_n) < \int_{S_n^\varepsilon}^{S_n} \frac{\exp^{[p-2]} (\log^{[q-1]} x)^\beta}{x}\, dx$$

which implies

$$\text{(2.3.6)} \qquad \log \mu(S_n) < \{\exp^{[p-2]} (\log^{[q-1]} S_n)^\beta\} \log S_n.$$

Now since, by (1.6.2),

$$M(S_n) < \mu(S_n) \{1 + 2\nu(S_n + \frac{S_n}{\nu(S_n)})\}$$

and

$$\nu(S_n + \frac{S_n}{\nu(S_n)}) < \nu(2S_n) < \exp^{[p-2]} (\log^{[q-1]} 2 S_n)^\beta ,$$

it follows that if $\lambda > 0$,

(2.3.7) $\quad \log \mu(S_n) \sim \log M(S_n)$ as $S_n \to \infty$.

From (2.3.6) and (2.3.7) we deduce that for $p = q = 2$,

$$(2.3.8) \quad \liminf_{n \to \infty} \frac{\log^{[2]} M(S_n)}{\log^{[2]} S_n} = \frac{\log^{[2]} \mu(S_n)}{\log^{[2]} S_n} \leq \beta + 1 ;$$

for $p = q \geq 3$,

$$(2.3.9) \quad \liminf_{n \to \infty} \frac{\log^{[p]} M(S_n)}{\log^{[p]} S_n} = \liminf_{n \to \infty} \frac{\log^{[p]} \mu(S_n)}{\log^{[p]} S_n} \leq \max(1,\beta),$$

and for $p > q$,

$$(2.3.10) \quad \liminf_{n \to \infty} \frac{\log^{[p]} M(S_n)}{\log^{[q]} S_n} = \liminf_{n \to \infty} \frac{\log^{[p]} \mu(S_n)}{\log^{[q]} S_n} \leq \beta .$$

Since (2.3.8), (2.3.9) and (2.3.10) hold for every $\beta > \theta$, it follows that $\lambda \leq P(\theta)$. This inequality is obvious for $\theta = \infty$. If $\lambda = 0$, in view of the inequality $\lambda \geq P(\theta)$, proved earlier, we have $P(\theta) = 0$. This completes the proof.

REMARK. This lemma generalizes a result of Shah and Ishaq [71] which was obtained when $(p,q) = (p,p)$ or $(p,p-1)$. For $(p,q) = (2,1)$ the lemma is due to Whittaker [95].

2.4. In this section we obtain a characterization of the (p,q)-order of an entire function $f(z)$ involving coefficients a_n's and exponents λ_n's of its Taylor series (2.1.1).

THEOREM 2.1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function of* (p,q)-*order* ρ , *then*

(2.4.1) $$\rho = P(L)$$

where

(2.4.2) $$L = \limsup_{k \to \infty} \frac{\log^{[p-1]} \lambda_k}{\log^{[q-1]} (\frac{1}{\lambda_k} \log |a_k|^{-1})} .$$

PROOF. By the definition of (p,q)-order, for any $\varepsilon > 0$ and for all $r > r_o = r_o(\varepsilon)$,

$$\log M(r) < \exp^{[p-2]} (\log^{[q-1]} r)^{\rho+\varepsilon}$$

Using Cauchy's estimate, this gives

(2.4.3) $$\log |a_k| < \exp^{[p-2]} (\log^{[q-1]} r)^{\rho+\varepsilon} - \lambda_k \log r$$

for all $r > r_o$ and for all k. For $(p,q) \neq (2,2)$, let

$$r = \exp^{[q-1]} (\log^{[p-2]} \frac{\lambda_k}{\rho+\varepsilon})^{1/(\rho+\varepsilon)} .$$

Then by (2.4.3), for all $k > k_o = k_o(r_o)$, and for $(p,q) \neq (2,2)$,

$$\log |a_k| < \frac{\lambda_k}{\rho+\varepsilon} - \lambda_k \exp^{[q-2]} (\log^{[p-2]} \frac{\lambda_k}{\rho+\varepsilon}).$$

Now for $p = q$, $\rho \geq 1$. Therefore, for $p = q \geq 3$, the above inequality gives

(2.4.4) $$\rho \geq \max (1, L),$$

whereas, for $p > q$, it gives

$$(2.4.5) \qquad \rho \geq L .$$

For $(p,q) = (2,2)$, let $r = \exp \{\lambda_k/(\rho + \varepsilon)\}^{1/(\rho-1+\varepsilon)}$. Then (2.4.3) gives, for all $k > k_o' = k_o'(r_o)$,

$$\log |a_k|^{-1/\lambda_k} > \left(\frac{\lambda_k}{\rho+\varepsilon}\right)^{1/(\rho-1+\varepsilon)} \left(\frac{\rho-1+\varepsilon}{\rho-1}\right)$$

and so

$$(2.4.6) \qquad \rho \geq 1 + \limsup_{k \to \infty} \frac{\log \lambda_k}{\log \left(\frac{1}{\lambda_k} \log |a_k|^{-1}\right)} .$$

(2.4.4), (2.4.5) and (2.4.6) together give $\rho \geq P(L)$. To prove the reverse inequality we note that $0 \leq L < \infty$. For any $\varepsilon > 0$, by the definition of L,

$$(2.4.7) \qquad |a_k| < \exp \{-\lambda_k \exp^{[q-2]} (\log^{[p-2]} \lambda_k)^{1/(L+\varepsilon)}\} ,$$

for all $k > k_o = k_o(\varepsilon)$. Since

$$M(r) \leq \sum_{k=0}^{\infty} |a_k| r^{\lambda_k} = A(k_o) + \sum_{k=k_o+1}^{S} |a_k| r^{\lambda_k} + \sum_{k=S+1}^{\infty} |a_k| r^{\lambda_k},$$

where $A(k_o)$ is a polynomial of degree at most k_o and $S=[\exp^{[p-2]}(\log^{[q-1]}2r)^{L+}$ by using (2.4.7) we get

$$M(r) < A(k_o) + \exp \{\exp^{[p-2]}(\log^{[q-1]}2r)^{L+\varepsilon} \times \log r\} \times$$

$$\times \sum_{k=0}^{\infty} \exp \{-\lambda_k \exp^{[q-2]} (\log^{[p-2]} \lambda_k)^{1/(L+\varepsilon)}\} + \sum_{k=0}^{\infty} 2^{-k} .$$

Since both the series occuring in the above expression are convergent, we have

$$\log^{[2]} M(r) < \exp^{[p-3]} (\log^{[q-1]} 2r)^{L+\varepsilon} + \log^{[2]} r + o(1)$$

and therefore $\rho \le P(L)$. This completes the proof of the theorem.

REMARK. The above theorem generalizes results of Sato [60] and Shah and Ishaq [71] which were obtained separately for $(p,q) = (p,1)$ and $(p,q) = (p,p)$ or $(p,p-1)$.

Next we have

THEOREM 2.2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function of* (p,q)*-order* ρ, *then*

(2.4.8) $$\rho \le P(L^*),$$

where,

(2.4.9) $$L^* = \limsup_{k\to\infty} \frac{\log^{[p-1]} \lambda_k}{\log^{[q-1]} \left(\frac{1}{\lambda_k - \lambda_{k-1}} \log |a_{k-1}/a_k|\right)}.$$

Further, equality holds in (2.4.8) *if* $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ *is a nondecreasing function of* k *for* $k > k_0$.

We require a lemma.

LEMMA 2.4. *Let* $\{\lambda_k\}_{k=0}^{\infty}$ *be an increasing sequence of positive integers and* $\{a_k\}_{k=0}^{\infty}$ *a sequence of complex numbers such that for some integer* $q \ge 1$, $|a_k|^{-1/\lambda_k} > \exp^{[q-1]}(1)$ *for* $k > k_0$. *Then for a pair* (p,q) *of positive integers such that* $p \ge q$, $p \ge 2$, $q \ge 1$,

$$(2.4.10)\quad \limsup_{k\to\infty} \frac{\log^{[p-1]} \lambda_k}{\log^{[q-1]} (\frac{1}{\lambda_k} \log |a_k|^{-1})} \leq \limsup_{k\to\infty} \frac{\log^{[p-1]} \lambda_k}{\log^{[q-1]} (\frac{1}{\lambda_k - \lambda_{k-1}} \log |a_{k-1}/a_k|)}$$

Further, equality holds in (2.4.10), if for $k > k_o$, $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ is a nondecreasing function tending to infinity with k.

PROOF. Let L^* be defined by (2.4.9). Since $|a_k|^{-1/\lambda_k} > \exp^{[q-1]}(1)$, we have $0 \leq L^* \leq \infty$. Since for $L^* = \infty$, (2.4.10) is trivially true we assume that $L^* < \infty$. Then for any $\varepsilon > 0$ and for all $m > N = N(\varepsilon)$, we get

$$\frac{\log^{[p-1]} \lambda_m}{\log^{[q-1]} (\frac{1}{\lambda_m - \lambda_{m-1}} \log |a_{m-1}/a_m|)} < L^* + \varepsilon,$$

which gives,

$$\log |a_{m-1}/a_m| > (\lambda_m - \lambda_{m-1}) \exp^{[q-2]} (\log^{[p-2]} \lambda_m)^{1/(L^*+\varepsilon)} .$$

Writing this inequality for $m = N+1, N+2, \ldots, k$ and adding all such inequalities, we get

$$(2.4.11)\quad \log|a_k|^{-1} > \log|a_N|^{-1} + \sum_{m=N+1}^{k} (\lambda_m - \lambda_{m-1}) \exp^{[q-2]} (\log^{[p-2]} \lambda_m)^{1/(L^*+\varepsilon)}$$

$$= \log |a_N|^{-1} + \lambda_k \exp^{[q-2]} (\log^{[p-2]} \lambda_k)^{1/(L^*+\varepsilon)} - J(\lambda_k) - \lambda_N \exp^{[q-2]} (\log^{[p-2]} \lambda_{N+1})^{1/(L^*+\varepsilon)}$$

where,

$$J(\lambda_k) = \int_{\lambda_{N+1}}^{\lambda_k} n(t) d\{\exp^{[q-2]} (\log^{[p-2]} t)^{1/(L^*+\varepsilon)}\}; \; n(t) = \lambda_m \text{ for } \lambda_m \leq t < \lambda_{m+1} .$$

$J(\lambda_k)$ satisfies

(2.4.12) $$J(\lambda_k) = \int_{\lambda_{N+1}}^{\lambda_k} n(t) \frac{E_{[q-2]}(\log^{[p-2]} t)^{1/(L^*+\varepsilon)}}{\Lambda_{[p-2]}(t)} dt$$

$$< \int_{\lambda_{N+1}}^{\lambda_k} \frac{t\, E_{[q-2]} (\log^{[p-2]} t)^{1/(L^*+\varepsilon)}}{\Lambda_{[p-2]}(t)} dt.$$

For $(p,q) = (2,1)$, by (2.4.12), $J(\lambda_k) < (\lambda_k - \lambda_{N+1})$, hence (2.4.11) gives

$$\log |a_k|^{-1} > O(1) + \frac{\lambda_k}{L^*+\varepsilon} \log \lambda_k - (\lambda_k - \lambda_N),$$

and therefore for $(p,q) = (2,1)$,

(2.4.13) $$\limsup_{k\to\infty} \frac{\lambda_k \log \lambda_k}{\log |a_k|^{-1}} \le L^* .$$

For $(p,q) = (2,2)$, by (2.4.12),

$$J(\lambda_k) < \frac{L^* + \varepsilon}{L^* + \varepsilon + 1} [\lambda_k^{(L^*+\varepsilon+1)/(L^*+\varepsilon)} - \lambda_{N+1}^{(L^*+\varepsilon+1)/(L^*+\varepsilon)}],$$

Hence by (2.4.11),

$$\log |a_k|^{-1} > O(1) + \frac{1}{L^*+\varepsilon+1} \lambda_k^{(L^*+\varepsilon+1)/(L^*+\varepsilon)},$$

which gives for $(p,q) = (2,2)$,

(2.4.14) $$\limsup_{k\to\infty} \frac{\log \lambda_k}{\log (\frac{1}{\lambda_k} \log|a_k|^{-1})} \le L^* .$$

Now for all pairs of integers (p,q) such that $p \geq 3$, we get from (2.4.12),

$$J(\lambda_k) < \exp^{[q-2]}(\log^{[p-2]}\lambda_k)^{1/(L^*+\varepsilon)} \{ (\int_{\lambda_{N+1}}^{[\sqrt{\lambda_k}]} + \int_{[\sqrt{\lambda_k}]}^{\lambda_k}) H(t)dt \},$$

where,

$$H(t) = \frac{t\ E_{[q-3]}(\log^{[p-2]}t)^{1/(L^*+\varepsilon)}}{\Lambda_{[p-2]}t} .$$

It is easily seen that if $p \geq 3$ then, for the index pair (p,q) such that either $p > q$ or $p = q$ and $L^* > 1$, $H(t)$ is a decreasing function of t and $H(t) \to 0$ as $t \to \infty$ while for $p = q$ and $L^* \leq 1$ $H(t)$ is an increasing function of t and $H(t) \to \infty$ as $t \to \infty$. Hence for pairs of integers (p,q) such that $p \geq 3$, $p > q$ or $p = q$ and $L^* > 1$,

$$J(\lambda_k) < \exp^{[q-2]}(\log^{[p-2]}\lambda_k)^{1/(L^*+\varepsilon)}[(\sqrt{\lambda_k}-\lambda_{N+1})H(\lambda_{N+1})+(\lambda_k-\sqrt{\lambda_k})H(\sqrt{\lambda_k})],$$

and, for pairs of integers (p,q) such that $p = q \geq 3$ and $L^* \leq 1$,

$$J(\lambda_k) < \exp^{[q-2]}(\log^{[p-2]}\lambda_k)^{1/(L^*+\varepsilon)}\ H(\lambda_k)\ (\lambda_k - \lambda_N) .$$

Using these estimates for $J(\lambda_k)$, we get, by (2.4.11), for all pairs of integers (p,q) such that $p \geq 3$

$$\log |a_k|^{-1} > O(1) + \lambda_k \exp^{[q-2]}(\log^{[p-2]}\lambda_k)^{1/(L^*+\varepsilon)}\ (1-O(1)).$$

Thus, for pairs of integers (p,q) such that $p \geq 3$,

$$\text{(2.4.15)} \qquad \limsup_{k \to \infty} \frac{\log^{[p-1]}\lambda_k}{\log^{[q-1]}(\frac{1}{\lambda_k}\log a_k^{-1})} \leq L^* .$$

(2.4.13), (2.4.14) and (2.4.15) give (2.4.10).

Now we prove the reverse inequality. Since $\psi(k)$ is a nondecreasing function of k for $k > k_o$, therefore for a fixed $N > k_o$ and all $k > N$, we have

$$\log |a_N/a_k| = \log |a_N/a_{N+1}| + \ldots + \log |a_{k-1}/a_k|$$

$$= (\lambda_{N+1} - \lambda_N) \log \psi(N) + \ldots + (\lambda_k - \lambda_{k-1}) \log \psi(k-1)$$

$$\leq (\lambda_k - \lambda_N) \log \psi(k-1).$$

Hence

$$\frac{\log^{[p-1]} \lambda_k}{\log^{[q-1]} (\frac{1}{\lambda_k} \log |a_k|^{-1})} \geq \frac{\log^{[p-1]} \lambda_k}{\{\log^{[q]} \psi(k-1)\} (1+o(1))} .$$

Thus, on proceeding to limits, we get

$$\limsup_{k \to \infty} \frac{\log^{[p-1]} \lambda_k}{\log^{[q-1]} (\frac{1}{\lambda_k} \log |a_k|^{-1})} \geq L^*$$

and the proof of the lemma is complete.

PROOF OF THEOREM 2.2. Since the hypothesis $|a_k|^{-1/\lambda_k} > \exp^{[q-1]} (1)$ is satisfied for the exponents $\{\lambda_k\}$ and coefficients $\{a_k\}$ of the series $\sum_{k=0}^{\infty} a_k z^{\lambda_k}$, we have by (2.4.10) that $L \leq L^*$, where L is given by (2.4.2). Now applying theorem 2.1, we get $\rho = P(L) \leq P(L^*)$. Further, if $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ is a nondecreasing function of k for $k > k_o$, then by the above lemma, $L = L^*$, which in view of Theorem 2.1 gives $\rho = P(L) = P(L^*)$. This completes the proof of Theorem 2.2.

COROLLARY. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function having* (p,q)-*order* ρ. *Let* $\{\lambda_{n_k}\}_{k=0}^{\infty}$ *be the principal indices and* $\rho(n_k)$ *be the jump points of the central index of* $f(z)$. *Then*

$$\rho = P(U),$$

where,

$$U = \limsup_{k \to \infty} \frac{\log^{[p-1]} \lambda_{n_k}}{\log^{[q]} \rho(n_k)} .$$

REMARK. Theorem 2.2 generalizes a result of Bajpai [6] which was obtained for the case $(p,q) = (p,1)$ and $\lambda_k = k$. For $(p,q) = (2,1)$ and $\lambda_k = k$ the theorem was obtained by Shah [63]. The theorem also generalizes a result of Awasthi [4] who obtained (2.4.1) for the case $(p,q) = (2,2)$ under the additional hypothesis $\lambda_k \sim \lambda_{k+1}$ as $k \to \infty$. For $(p,q) = (2,1)$ and $\lambda_k = k$, the corollary was obtained by Gray and Shah [26] by a different technique.

2.5. We observe that a result analogous to (2.4.1) does not hold always for the lower (p,q)-order of an entire function, i.e., there exists an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ of lower (p,q)-order λ such that $\lambda = P(X)$ does not hold, where

$$X = \liminf_{k \to \infty} \frac{\log^{[p-1]} \lambda_k}{\log^{[q-1]} \left(\frac{1}{\lambda_k} \log |a_k|^{-1}\right)} .$$

This can be seen by the following example.

EXAMPLE. Let, for $q \geq 2$,

$$f(z) = \sum_{k=1}^{\infty} \exp\{-k \exp^{[q-2]} (\log^{[p-2]} k)^{1/3}\} z^{2k} +$$

$$+ \sum_{k=1}^{\infty} \exp\{-k \exp^{[q-2]} (\log^{[p-2]} k)^{1/2}\} z^{k} = f_1(z) + f_2(z) \text{ (say)},$$

and set,

$$\begin{matrix} \theta \\ \theta \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \nu(r,f_1)}{\log^{[q]} r}.$$

Then, $\nu(r,f_1) = 2k$ for $\psi(k-1) \leq r < \psi(k)$, where

$$\psi(k) = [\exp\{-k \exp^{[q-2]} (\log^{[p-2]} k)^{1/3} + (k+1)\exp^{[q-2]} (\log^{[p-2]} (k+1))^{1/3}\}]^{1/2}.$$

For $q \geq 2$, $\log^{[q]} \psi(k) \simeq \frac{1}{3} \log^{[p-1]} k$ as $k \to \infty$. Hence,

$$\frac{\log^{[p-1]} 2k}{\log^{[q]} \{\psi(k)\}} \sim 3 \text{ as } k \to \infty.$$

This shows that $f_1(z)$ is of index pair (p,q) and its (p,q)-order and lower (p,q)-order are $P(3)$.

Further, for sufficiently large r,

$$M(r,f_1) \leq M(r,f) = M(r,f_1) + M(r,f_2) \leq 2M(r,f_1),$$

therefore it follows that $f(z)$ is of index pair (p,q) and its (p,q)-order and lower (p,q)-order are $P(3)$. But

$$X = \liminf_{k \to \infty} \frac{\log^{[p-1]} k}{\log^{[q-1]} (\frac{1}{k} \log |a_k|^{-1})} = 2 .$$

However, we have the following theorem:

THEOREM 2.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function having lower* (p,q)*-order* λ *and let* $\{n_k\}_{k=1}^{\infty}$ *be an increasing sequence of natural numbers, then*

(2.5.1) $$\lambda \geq P_{\chi}(\ell)$$

where ,

(2.5.2) $$\chi \equiv \chi(\{n_k\}) = \liminf_{k \to \infty} \frac{\log \lambda_{n_{k-1}}}{\log \lambda_{n_k}}$$

and

(2.5.3) $$\ell \equiv \ell(\{n_k\}) = \liminf_{k \to \infty} \frac{\log^{[p-1]} \lambda_{n_{k-1}}}{\log^{[q-1]} (\frac{1}{\lambda_{n_k}} \log |a_{n_k}|^{-1})} .$$

PROOF. We note that $0 \leq \ell \leq \infty$. First let $0 < \ell < \infty$. Then, for $\ell > \varepsilon > 0$, and $k > k_o = k_o(\varepsilon)$,

(2.5.4) $$|a_{n_k}| > \exp \{-\lambda_{n_k} \exp^{[q-2]} (\log^{[p-2]} \lambda_{n_{k-1}})^{1/(\ell-\varepsilon)} \} .$$

Since addition of a polynomial does not affect the lower (p,q)-order of $f(z)$ we can assume that (2.5.4) holds for all k. Let $(p,q) \neq (2,2)$ and choose

(2.5.5) $$r_k = 2 \exp^{[q-1]} (\log^{[p-2]} \lambda_{n_{k-1}})^{1/(\ell-\varepsilon)} \quad \text{for } k = 2,3,\ldots .$$

If $r_k \leq r \leq r_{k+1}$, then (2.5.4) gives for all $r > r_o = r_o(p)$ and for $(p,q) \neq (2,2)$,

$$\begin{aligned}\log M(r) &\geq \log |a_{n_k}| + \lambda_{n_k} \log r \\ &\geq -\lambda_{n_k} \exp^{[q-2]} (\log^{[p-2]} \lambda_{n_{k-1}})^{1/(\ell-\varepsilon)} + \lambda_{n_k} \log r_k \\ &> \lambda_{n_k} \log 2 \\ &= \{\exp^{[p-2]} (\log^{[q-1]} \tfrac{r_{k+1}}{2})^{\ell-\varepsilon}\} \log 2 \\ &\geq \{\exp^{[p-2]} (\log^{[q-1]} \tfrac{r}{2})^{\ell-\varepsilon}\} \log 2.\end{aligned}$$

Therefore, for sufficiently large values of r, and for $(p,q) \neq (2,2)$

$$\log^{[p]} M(r) > (\ell-\varepsilon) \log^{[q]} \frac{r}{2} + o(1).$$

For $p = q$ we have $\lambda \geq 1$, hence this inequality gives, for $p = q \geq 3$,

(2.5.6) $$\liminf_{r \to \infty} \frac{\log^{[p]} M(r)}{\log^{[p]} r} \geq \max (1,\ell),$$

and for $p > q$,

(2.5.7) $$\liminf_{r \to \infty} \frac{\log^{[p]} M(r)}{\log^{[q]} r} \geq \ell .$$

Next for $(p,q) = (2,2)$, we observe that $1 \leq \ell < \infty$ and choose

(2.5.8) $$r_k = \exp (2 \lambda_{n_{k-1}}^{1/(\ell-\varepsilon)}) \text{ for } k = 2,3,\ldots .$$

If $r_k \leq r \leq r_{k+1}$, proceeding as above, we get

$$\log M(r) > \lambda_{n_k} \lambda_{n_{k-1}}^{1/(\ell-\varepsilon)} ,$$

which gives

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$$\frac{\log^{[2]} M(r)}{\log^{[2]} r} > \frac{\log \lambda_{n_k}}{\log^{[2]} r_{k+1}} + \frac{1}{\ell-\varepsilon}\,\frac{\log \lambda_{n_{k-1}}}{\log^{[2]} r_{k+1}} .$$

Substituting the value of $\log^{[2]} r_{k+1}$ from (2.5.8) and proceeding to limits we get for $(p,q) = (2,2)$

(2.5.9) $$\lambda \geq \ell + \chi.$$

(2.5.6), (2.5.7) and (2.5.9) together prove (2.5.1). When $\ell = 0$, (2.5.1) is trivially true. ℓ cannot be infinite, since in that case above arguments can be repeated with an arbitrarily large number in place of $(\ell - \varepsilon)$ to give $\lambda = \infty$ which is not possible for a function having index pair (p,q). This completes the proof of the theorem.

REMARKS. (i) From (2.4.1) and (2.5.1) it follows that for an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$

$$P_{\chi_0}(\ell_0) \leq \lambda \leq \rho = P(L)$$

where $\chi_0 = \chi(\{k\})$ and $\ell_0 = \ell(\{k\})$. Hence if (a) $\log^{[p-1]} \lambda_{k-1} \sim \log^{[p-1]} \lambda_k$ as $k \to \infty$ and (b) $S \equiv \lim_{k\to\infty} \dfrac{\log^{[p-1]} \lambda_k}{\log^{[q-1]}\left(\frac{1}{\lambda_k} \log |a_k|^{-1}\right)}$ exists then $f(z)$ is of regular (p,q) growth and in that case $\rho = \lambda = P(S)$.

(ii) For $(p,q) = (2,1)$ this theorem is due to Juneja [34].

THEOREM 2.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function having lower (p,q)-order λ and let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers, then*

(2.5.10) $\lambda \geq P_\chi(\ell^*)$

where

(2.5.11) $\ell^* \equiv \ell^*(\{n_k\}) = \liminf_{k\to\infty} \dfrac{\log^{[p-1]} \lambda_{n_{k-1}}}{\log^{[q-1]}\left(\frac{1}{\lambda_{n_k}-\lambda_{n_{k-1}}} \log |a_{n_{k-1}}/a_{n_k}|\right)}$.

PROOF. (2.5.10) obviously holds when ℓ^* is negative on zero, therefore it is sufficient to consider that $0 < \ell^* \leq \infty$. First let $0 < \ell^* < \infty$. For any ε such that $\ell^* > \varepsilon > 0$ and for all $k > N = N(\varepsilon)$, by (2.5.11), we have

$$|a_{n_{k-1}}/a_{n_k}| < \exp\{(\lambda_{n_k} - \lambda_{n_{k-1}})\exp^{[q-2]} (\log^{[p-2]} \lambda_{n_{k-1}})^{1/(\ell^*-\varepsilon)}\}$$

which gives

(2.5.12) $|a_{n_k}| > |a_{n_N}| \prod_{m=N+1}^{k} \exp\{-(\lambda_{n_m}-\lambda_{n_{m-1}})\exp^{[q-2]}(\log^{[p-2]}\lambda_{n_{m-1}})^{1/(\ell^*-\varepsilon)}\}$.

Let, for $(p,q) \neq (2,2)$,

$$r_k = 2 \exp^{[q-1]} (\log^{[p-2]} \lambda_{n_{k-1}})^{1/(\ell^*-\varepsilon)}$$

and for $(p,q) = (2,2)$

$$r_k = \exp(2\lambda_{n_{k-1}}^{1/(\ell^*-\varepsilon)}) , \; k = 2,3,\dots .$$

Then, for $r_k \leq r \leq r_{k+1}$, by (2.5.12),

$$\log M(r) \geq \log |a_{n_k}| + \lambda_{n_k} \log r_k$$

$$> \log |a_{n_N}| - \sum_{m=N+1}^{k} (\lambda_{n_m} - \lambda_{n_{m-1}})\exp^{[q-2]}(\log^{[p-2]}\lambda_{n_{m-1}})^{1/(\ell^*-\varepsilon)}$$

$$+ \lambda_{n_k} \log r_k.$$

Or,

(2.5.13)

$$\log M(r) > \log |a_{n_N}| - \lambda_{n_k} \exp^{[q-2]}(\log^{[p-2]}\lambda_{n_{k-1}})^{1/(\ell^*-\varepsilon)} + \lambda_{n_k} \log r_k$$

For $(p,q) \neq (2,2)$, (2.5.13) gives

$$\log M(r) > \log |a_{n_N}| + \lambda_{n_k} \log 2$$

$$\geq \log |a_{n_N}| + \{\exp^{[p-2]} (\log^{[q-1]} \tfrac{r}{2})^{\ell^*-\varepsilon}\} \log 2$$

and so

(2.5.14) $$\lambda \geq P(\ell^*).$$

For $(p,q) = (2,2)$, (2.5.13) gives

$$\frac{\log^{[2]} M(r)}{\log^{[2]} r} > \frac{\log \lambda_{n_k}}{\log^{[2]} r_{k+1}} + \frac{1}{\ell^*-\varepsilon} \frac{\log \lambda_{n_{k-1}}}{\log^{[2]} r_{k+1}},$$

hence, we get for $(p,q) = (2,2)$

(2.5.15) $$\lambda \geq \chi + \ell^* .$$

(2.5.14) and (2.5.15) give (2.5.10) for the case $0 \leq \ell^* < \infty$. ℓ^* cannot be infinite since in that case above arguments can be repeated with an arbitrarily large number in place of $(\ell^* - \varepsilon)$ to give $\lambda = \infty$ which is not possible for a function of index pair (p,q). Hence the proof of the theorem is complete.

REMARKS. (i) From (2.4.8) and (2.5.10) it follows that if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is an entire function of (p,q)-order ρ and lower (p,q)-order λ, then

$$P_{\chi_o}(\ell_o^*) \leq \lambda \leq \rho \leq P(L^*)$$

where $\chi_o = \chi(\{k\})$ and $\ell_o^* = \ell^*(\{k\})$. Hence if (a) $\log^{[p-1]}\lambda_{k-1} \sim \log^{[p-1]}\lambda_k$ as $k \to \infty$ (b) $S^* = \lim_{k\to\infty} \dfrac{\log^{[p-1]}\lambda_k}{\log^{[q-1]}\left(\frac{1}{\lambda_k-\lambda_{k-1}}\log|a_{k-1}/a_k|\right)}$ exists, then f(z) is of regular (p,q)-growth and in that case $\rho = \lambda = P(S^*)$.

(ii) For $(p,q) = (2,1)$, Theorem 2.4 is due to Juneja and Kapoor [35].

THEOREM 2.5. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function of lower (p,q)-order* λ *such that* $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ *is a nondecreasing function of k for* $k > k_o$, *then for* $(p,q) \neq (2,2)$

$$\lambda = P(\ell_o) \tag{2.5.16}$$

where

$$\ell_o = \liminf_{k\to\infty} \frac{\log^{[p-1]}\lambda_{k-1}}{\log^{[q-1]}\left(\frac{1}{\lambda_k}\log|a_k|^{-1}\right)}. \tag{2.5.17}$$

(2.5.16) *holds for* $(p,q) = (2,2)$ *also if further* $\log\lambda_{k-1} \sim \log\lambda_k$ *as* $k \to \infty$.

PROOF. In view of Theorem 2.3, it is sufficient to prove that $\lambda \leq P(\ell_o)$. Since by hypothesis, $\psi(k)$ is a nondecreasing function of k for $k > k_o$, we have $\psi(k) > \psi(k-1)$ for infinitely many values of k because if it is not so, then $\psi(k) = \psi(k+1) = \ldots$ ad infinitum, for $k > k_o$ (say) and so the

radius of convergence of $\sum_{k=0}^{\infty} a_k z^{\lambda_k}$ would be finite and consequently $f(z)$ ceases to be an entire function. $\psi(k) \to \infty$ as $k \to \infty$.

When $\psi(k) > \psi(k-1)$, the term $a_k z^{\lambda_k}$ becomes maximum term and we have

$$\mu(r) = |a_k|\, r^{\lambda_k}, \quad \nu(r) = \lambda_k \text{ for } \psi(k-1) \leq r < \psi(k).$$

Now let θ be as in (2.3.1) and first assume that $\theta > 0$. For any such that $\theta > \varepsilon > 0$ and for all $r > r_o = r_o(\varepsilon)$, we have

$$\nu(r) > \exp^{[p-2]} (\log^{[q-1]} r)^{\theta-\varepsilon}. \tag{2.5.18}$$

Let $a_{k_1} z^{\lambda_{k_1}}$ and $a_{k_2} z^{\lambda_{k_2}}$ $(k_1 > k_o,\ \psi(k_1-1) > r_o)$ be two consecutive maximum terms so that $k_1 \leq k_2 - 1$. Let $k_1 < k \leq k_2$. Since $a_{k_1} z^{\lambda_{k_1}}$ is maximum term, we have $\nu(r) = \lambda_{k_1}$ for $\psi(k_1-1) \leq r < \psi(k_1)$. Hence, for r in this interval

$$\lambda_{k_1} = \nu(r) > \exp^{[p-2]} (\log^{[q-1]} r)^{\theta-\varepsilon}.$$

Further, since

$$\psi(k_1) = \psi(k_1+1) = - - - - = \psi(k-1),$$

we have

$$\lambda_{k-1} \geq \lambda_{k_1} > \exp^{[p-2]} \{\log^{[q-1]} (\psi(k-1)-c)\}^{\theta-\varepsilon}, \tag{2.5.19}$$

where $c = \min\ [1, (\psi(k_1) - \psi(k_1 - 1))/2]$. Now, since $\psi(k)$ is nondecreasing

$$\log|a_{k_0}/a_{k_0+1}| + \ldots + \log|a_{k-1}/a_k| = \log|a_{k_0}/a_k| \leq (\lambda_k - \lambda_{k_0})\log\psi(k-1),$$

and therefore for sufficiently large k,

$$\frac{\log^{[q-1]}\left(\frac{1}{\lambda_k}\log|a_k|^{-1}\right)}{\log^{[p-1]}\lambda_{k-1}} \leq \frac{1}{\theta-\varepsilon}.$$

On proceeding to limits this inequality gives

$$\theta \leq \ell_0. \tag{2.5.20}$$

Combining (2.3.4) and (2.5.20), we get $\lambda = P(\theta) \leq P(\ell_0)$. Hence the proof the theorem is complete.

REMARK. Theorem 2.5 generalizes a result of Juneja and Singh [38] which was obtained for the case $(p,q) = (2,1)$. It also includes a result of Shah and Ishaq [71] which was obtained for $(p,q) = (p,p)$ or $(p,p-1)$ and $\lambda_k = k$.

Next we have

THEOREM 2.6. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function having lower* (p,q)*-order* λ *such that* $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)}$ *forms a nondecreasing function of* k *for* $k > k_0$, *then for* $(p,q) \neq (2,2)$

$$\lambda = P(\ell_0^*) \tag{2.5.21}$$

where

$$\ell_o^* = \liminf_{k\to\infty} \frac{\log^{[p-1]} \lambda_{k-1}}{\log^{[q-1]} \left(\frac{1}{\lambda_k-\lambda_{k-1}} \log|a_{k-1}/a_k|\right)} .$$

(2.5.21) <u>holds for</u> $(p,q) = (2,2)$ <u>also if further</u> $\log \lambda_{k-1} \sim \log \lambda_k$ <u>as</u> $k \to \infty$.

We need the following lemma.

<u>LEMMA</u> 2.5. <u>Let</u> $\{\lambda_{n_k}\}_{k=0}^{\infty}$ <u>be an increasing sequence of positive integers and let</u> $\{a_n\}_{n=0}^{\infty}$ <u>be a sequence of nonzero complex numbers such that for some integer</u> $q \geq 1$, $|a_{n_k}|^{-1/\lambda_{n_k}} > \exp^{[q-1]}(1)$ <u>for</u> $k > k_1$, <u>then for a pair</u> (p,q) <u>of positive integers such that</u>, $p \geq q \geq 1$, $p \geq 2$,

$$(2.5.22)\quad \liminf_{k\to\infty} \frac{\log^{[p-1]} \lambda_{n_{k-1}}}{\log^{[q-1]}\left(\frac{1}{\lambda_{n_k}} \log |a_{n_k}|^{-1}\right)} \geq \liminf_{k\to\infty} \frac{\log^{[p-1]} \lambda_{n_{k-1}}}{\log^{[q-1]}\left(\frac{1}{\lambda_{n_k}-\lambda_{n_{k-1}}} \log|a_{n_{k-1}}/a_{n_k}|\right)}$$

<u>Further, equality holds in</u> (2.5.22) <u>if</u> for $k > k_o$, $\psi(n_k) \equiv |a_{n_k}/a_{n_{k+1}}|^{1/(\lambda_{n_{k+1}}-\lambda_{n_k})}$ <u>is a nondecreasing function tending to infinity with</u> k.

<u>PROOF</u>. Let ℓ^* be defined by (2.5.11). If ℓ^* is negative or zero then (2.5.22) obviously holds. Hence we assume that $\ell^* > 0$. For $\ell^* > \varepsilon > 0$ and for all $k > N = N(\varepsilon)$,

$$|a_{n_{k-1}}/a_{n_k}| < \exp\{(\lambda_{n_k}-\lambda_{n_{k-1}})\exp^{[q-2]}(\log^{[p-2]}\lambda_{n_{k-1}})^{1/(\ell^*-\varepsilon)}\} ,$$

which gives

$$|a_{n_k}| > |a_{n_N}| \prod_{m=N+1}^{k} \exp\{-(\lambda_{n_m}-\lambda_{n_{m-1}})\exp^{[q-2]}(\log^{[p-2]}\lambda_{n_{m-1}})^{1/(\ell^*-\varepsilon)}\} .$$

Or,

$$\log |a_{n_k}|^{-1} < \log |a_{n_N}|^{-1} + \sum_{m=N+1}^{k} (\lambda_{n_m} - \lambda_{n_{m-1}}) \exp^{[q-2]} (\log^{[p-2]} \lambda_{n_{m-1}})^{1/(\ell^*-\varepsilon)}$$

$$< \log |a_{n_N}|^{-1} + \lambda_{n_k} \exp^{[q-2]} (\log^{[p-2]} \lambda_{n_{k-1}})^{1/(\ell^*-\varepsilon)} .$$

Thus,

$$\log^{[q-1]} \left(\frac{1}{\lambda_{n_k}} \log |a_{n_k}|^{-1}\right) < o(1) + \frac{1}{\ell^*-\varepsilon} \log^{[p-1]} \lambda_{n_{k-1}} ,$$

which gives

$$\ell^* \leq \liminf_{k \to \infty} \frac{\log^{[p-1]} \lambda_{n_{k-1}}}{\log^{[q-1]} \left(\frac{1}{\lambda_{n_k}} \log |a_{n_k}|^{-1}\right)} .$$

Next we prove the reverse inequality. Since $\psi(n_k)$ is nondecreasing function of k for $k > k_o$, we have $0 \leq \ell^* \leq \infty$. When $\ell^* = \infty$, the inequality reverse to (2.5.22) is trivially true. Hence assume that $\ell^* < \infty$. By (2.5.11), for any $\varepsilon > 0$,

$$\psi(n_{k-1}) > \exp^{[q]} \left(\frac{1}{\ell^*+\varepsilon} \log^{[p-1]} \lambda_{n_{k-1}}\right)$$

for $k = k_1, k_2, \ldots, k_p, \ldots \to \infty$. Let $k_\ell > k^* = \max(k_o, k_1)$.

Since, $\psi(n_k)$ is nondecreasing function of k for $k > k_o$,

$$\log |a_{n_k}|^{-1} = \log|a_{n_{k_\ell}}|^{-1} + \log|a_{n_{k_\ell}}/a_{n_{k_\ell+1}}| + \ldots + \log|a_{n_{k_p}}/a_{n_{k_p+1}}| + \ldots + \log a_{n_{k-1}}/|a_{n_k}|$$

$$= \log|a_{n_{k_\ell}}|^{-1} + (\lambda_{n_{k_\ell+1}} - \lambda_{n_{k_\ell}}) \log \psi(n_{k_\ell}) + \ldots + (\lambda_{n_k} - \lambda_{n_{k-1}}) \log \psi(n_{k-1})$$

$$> \log |a_{n_{k_\ell}}|^{-1} + (\lambda_{n_{k_p}} - \lambda_{n_{k_p-1}}) \log \psi(n_{k_p-1}) + \ldots + (\lambda_{n_k} - \lambda_{n_{k-1}}) \log \psi(n_{k-1})$$

$$> (\lambda_{n_k} - \lambda_{n_{k_p-1}}) \log \psi(n_{k_p-1}) > (\lambda_{n_k} - \lambda_{n_{k_p-1}}) \exp^{[q-1]} \left(\frac{1}{\ell^*+\varepsilon} \log^{[p-1]} \lambda_{n_{k_p-1}}\right)$$

Evidently, for every k_p, there is a positive integer k_r such that

$\lambda_{n_{k_p-1}}/\lambda_{n_{k_r}} \leq \varepsilon < \lambda_{n_{k_p-1}}/\lambda_{n_{k_r-1}}$. Thus, we get

$$\frac{\log^{[q-1]}\left(\frac{1}{\lambda_{n_{k_r}}}\log|a_{n_{k_r}}|^{-1}\right)}{\log^{[p-1]}\lambda_{n_{k_r-1}}} > \frac{1}{\ell^*+\varepsilon}(1+o(1)).$$

Hence,

$$\ell^* \geq \liminf_{k\to\infty} \frac{\log^{[p-1]}\lambda_{n_{k-1}}}{\log^{[q-1]}\left(\frac{1}{\lambda_{n_k}}\log|a_{n_k}|^{-1}\right)}.$$

This completes the proof of the lemma.

PROOF OF THEOREM 2.6. Since the hypotheses of Lemma 2.5 are satisfied for exponents $\{\lambda_k\}$ and coefficients $\{a_k\}$ of the series $\sum_{k=0}^{\infty} a_k z^{\lambda_k}$, we have $\ell_o = \ell_o^*$ where ℓ_o is defined by (2.5.17). Now, using Theorem 2.5, we get $\lambda = P(\ell_o) = P(\ell_o^*)$. This proves the theorem.

COROLLARY. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function of lower (p,q)-order* λ. *Let* $\{\lambda_{n_k}\}_{k=0}^{\infty}$ *be the principal indices and* $\rho(n_k)$ *be jump points of the central index of* $f(z)$. *Then, for* $(p,q) \neq (2,2)$

(2.5.23) $$\lambda = P(V),$$

where

$$V = \liminf_{k\to\infty} \frac{\log^{[p-1]}\lambda_{n_{k-1}}}{\log^{[q]}\rho(n_k)}.$$

(2.5.23) holds for $(p,q) = (2,2)$ also, if further $\log \lambda_{n_{k-1}} \sim \log \lambda_{n_k}$ as $k \to \infty$.

REMARKS. (i) Theorem 2.6 can alternatively be proved by using Theorem 2.4. Thus, since $\psi(k)$ is a nondecreaing function of k for $k > k_0$, proceeding as in the proof of Theorem 5 we get

$$\lambda_{k-1} > \exp^{[p-2]} \{\log^{[q-1]} (\psi(k-1)-c)\}^{\theta-\varepsilon}$$

where θ is defined by (2.3.1) and c is a constant less than or equal to 1. Putting the value of $\psi(k-1)$ and proceeding to limits after some simple calculations, we get $\theta \leq \ell_0^*$. Now using Lemma 2.3, the last inequality gives $\lambda = P(\theta) \leq P(\ell_0^*)$. But by Theorem 2.4 we have $\lambda \geq P_\chi(\ell_0^*)$, hence (2.5.21) follows.

(ii) An alternative proof of Theorem 2.4 can be given by using Lemma 2.5 and Theorem 2.3. Thus, since f(z) is an entire function $|a_{n_k}|^{-1/\lambda_{n_k}} > \exp^{[q-1]}(1)$ for $k > k_0$, hence by (2.5.22), $\ell^* \leq \ell$, where ℓ and ℓ^* are defined by (2.5.3) and (2.5.11) respectively. Now using Theorem 2.3, we get $\lambda = P_\chi(\ell) \geq P_\chi(\ell^*)$ which proves (2.5.10).

(iii) The above corollary generalizes a result of Gray and Shah [26] which was obtained by a different technique for $(p,q) = (2,1)$ and $\lambda_k = k$.

Now we prove the main theorem of this section. This gives a complete coefficient characterization for the lower order of those entire functions whose index pair is other than (2,2). For $(p,q) = (2,2)$, we obtain the coefficient equivalent of the lower (2,2)-order, under the additional condition $\log \lambda_{n_{k-1}} \sim \log \lambda_{n_k}$ as $k \to \infty$, $\{\lambda_{n_k}\}_{k=0}^{\infty}$ being the sequence of principal indices of the entire function under consideration.

THEOREM 2.7. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of lower (p,q)-order λ. Then for $(p,q) \neq (2,2)$

$$(2.5.24) \qquad \lambda = \max_{\{n_k\}} [P_\chi(\ell)]$$

$$(2.5.25) \qquad = \max_{\{n_k\}} [P_\chi(\ell^*)]$$

where χ, ℓ and ℓ^* are defined by (2.5.2), (2.5.3) and (2.5.11) respectively and maximum in (2.5.24) and (2.5.25) is taken over all increasing sequences $\{n_k\}$ of natural numbers. Further, if $\{\lambda_{n_k}\}$ are the principal indices of $f(z)$ such that $\log \lambda_{n_{k-1}} \sim \log \lambda_{n_k}$ as $k \to \infty$, then (2.5.24) and (2.5.25) hold for $(p,q) = (2,2)$ also.

PROOF. Consider the function $g(z) = \sum_{k=0}^{\infty} a_{n_k} z^{\lambda_{n_k}}$, where $\{\lambda_{n_k}\}_{k=0}^{\infty}$ are the principal indices of $f(z)$. It is easily seen that $g(z)$ is an entire function and that for any z, $f(z)$ and $g(z)$ have the same maximum term. Hence by (2.3.3) and (2.3.4), (p,q)-order and lower (p,q)-order of $g(z)$ are the same as those of $f(z)$. Thus $g(z)$ is of lower (p,q)-order λ. Further, since $\psi(n_k) \equiv |a_{n_k}/a_{n_{k+1}}|^{1/(\lambda_{n_{k+1}}-\lambda_{n_k})}$ is a nondecreasing function of k and for $(p,q) = (2,2)$ $\log \lambda_{n_{k-1}} \sim \log \lambda_{n_k}$ as $k \to \infty$, $g(z)$ satisfies the hypotheses of Theorems 2.5 and 2.6. Hence by (2.5.16) and (2.5.21)

$$(2.5.26) \qquad \lambda = P_{\chi(\{n_k\})}(\ell(\{n_k\})) = P_{\chi(\{n_k\})}(\ell^*(\{n_k\})).$$

But by Theorems 2.3 and 2.4

$$(2.5.27) \qquad \lambda \geq \max_{\{n_h\}} [P_{\chi(\{n_h\})}(\ell(\{n_h\}))]$$

and

$$(2.5.28) \qquad \lambda \geq \max_{\{n_h\}} [P_{\chi(\{n_h\})}(\ell^*(\{n_h\}))].$$

Now combining (2.5.26), (2.5.27) and (2.5.28), we get (2.5.24) and (2.5.25). Hence the theorem.

COROLLARY. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function of* (p,q)*-order* ρ *and lower* (p,q)*-order* λ. *Then for* $p > q$

$$(2.5.29) \qquad \lambda \leq \rho \liminf_{k \to \infty} \frac{\log^{[p-1]} \lambda_{k-1}}{\log^{[p-1]} \lambda_k}.$$

(2.5.29) *holds for* $p = q \geq 3$ *also if* ℓ *defined by* (2.5.3) *is greater than or equal to* 1.

By (2.5.24) and (2.4.1), if $p > q$ or if $p = q \geq 3$ and $\ell \geq 1$, then

$$\lambda \leq \left(\limsup_{k \to \infty} \frac{\log^{[p-1]}\lambda_k}{\log^{[q-1]}\left(\frac{1}{\lambda_k} \log |a_k|^{-1}\right)}\right) \left(\max_{\{n_k\}} \left[\liminf_{k \to \infty} \frac{\log^{[p-1]}\lambda_{n_{k-1}}}{\log^{[p-1]}\lambda_{n_k}}\right]\right)$$

$$= \rho \liminf_{k \to \infty} \frac{\log^{[p-1]}\lambda_{k-1}}{\log^{[p-1]}\lambda_k}.$$

REMARKS. (i) (2.5.24) generalizes a result of Juneja [34] and (2.5.25) generalizes a result contained in [35]. Both of these results were obtained for the case $(p,q) = (2,1)$.

(ii) The corollary generalizes a result of Whittaker [95] which was obtained for the case $(p,q) = (2,1)$.

CHAPTER 3

(p,q)-TYPE AND LOWER (p,q)-TYPE OF AN ENTIRE FUNCTION

3.1. In this chapter we continue the study of our classification scheme as introduced in Chapter 2. In Section 2.2 we have seen that if two entire functions are of the same (p,q)-order ρ, then (2.2.1) does not give any precise estimate about their relative growth. Thus, for the specification of rate of growth of such functions we introduce the concepts of (p,q)-type and lower (p,q)-type as follows.

DEFINITION 3.1. *An entire function* $f(z)$ *of* (p,q)-*order* ρ *is said to be of* (p,q)-*type* T *and lower* (p,q)-*type* t, *if*

$$(3.1.1)\qquad \begin{matrix} T \\ t \end{matrix} \equiv \begin{matrix} T(p,q) \\ t(p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho}}, \quad (0 \le t \le T \le \infty).$$

An entire function of regular (p,q)-growth is said to be of perfectly regular (p,q)-growth if $T(p,q) = t(p,q) < \infty$.

The (p,q)-growth numbers are defined to facilitate the study of growth of the central index of an entire function. Thus we have

DEFINITION 3.2. *An entire function* $f(z)$ *of* (p,q)-*order* ρ *is said to have* (p,q)-*growth number* μ *and lower* (p,q)-*growth number* δ, *if*

$$(3.1.2)\qquad \begin{matrix} \mu\equiv\mu(p,q) \\ \delta\equiv\delta(p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \nu(r)}{(\log^{[q-1]} r)^{\rho-A}}$$

where $\nu(r)$ *is the central index of* $f(z)$ *for* $|z| = r$, *and*

(3.1.3) $\quad A = 1$ *for* $(p,q) = (2,2)$ *and* $A = 0$ *otherwise.*

In the following sections we obtain coefficient equivalents of the constants defined in (3.1.1) and (3.1.2) and find some relations involving (p,q)-type, lower (p,q)-type and ratio of the two consecutive coefficients of the entire series (2.1.1).

The notations used in this chapter are the same as those given in section 2.2. Also, we shall assume throughout this chapter that the entire function $f(z)$ is represented by the power series (2.1.1).

3.2. In this section we obtain a complete coefficient characterization of (p,q)-type of an entire function. Thus we have,

THEOREM 3.1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function having* (p,q)*-order* ρ *and* (p,q)*-type* T, *then*

$$(3.2.1) \qquad T/M = \limsup_{k \to \infty} \frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}},$$

where A *is defined by* (3.1.3) *and*

$$(3.2.2) \qquad M = \frac{(\rho-1)^{\rho-1}}{\rho^{\rho}} \quad \text{if} \quad (p,q) = (2,2)$$

$$= \frac{1}{e\rho} \qquad \text{if} \quad (p,q) = (2,1)$$

$$= 1 \qquad \text{for all other index pairs } (p,q).$$

PROOF: First assume that $T < \infty$. By (3.1.1), given $\varepsilon > 0$, we have, for all $r > r_o = r_o(\varepsilon)$,

$$\log M(r) < \exp^{[p-2]} \{(T+\varepsilon)\ (\log^{[q-1]} r)^{\rho}\}.$$

Using Cauchy's estimate this gives for all k and all $r > r_o$,

$$(3.2.3) \qquad \log |a_k| < \exp^{[p-2]} \{(T+\varepsilon)\ (\log^{[q-1]} r)^{\rho}\} - \lambda_k \log r.$$

Let,

$$r = \exp^{[q-1]} \left(\frac{1}{T+\varepsilon} \log^{[p-2]} \frac{\lambda_k}{\rho}\right)^{1/(\rho-A)}.$$

Then (3.2.3) gives for all $k > k_o = k_o(r_o)$ and for $(p,q) \neq (2,2)$

$$\log |a_k| < \frac{\lambda_k}{\rho} - \lambda_k \exp^{[q-2]} \left(\frac{1}{T+\varepsilon} \log^{[p-2]} \frac{\lambda_k}{\rho}\right)^{1/\rho}$$

and so, for $(p,q) \neq (2,2)$,

$$(3.2.4) \qquad T/M \geq \limsup_{k \to \infty} \frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho}}.$$

For $(p,q) = (2,2)$, using (3.2.3) once again, we get for all $k > k_o$,

$$\log |a_k|^{-1/\lambda_k} > \frac{(\rho-1)}{\rho} \left\{\frac{\lambda_k}{\rho\ (T+\varepsilon)}\right\}^{1/(\rho-1)}$$

and so for $(p,q) = (2,2)$

$$(3.2.5) \qquad T \frac{\rho^{\rho}}{(\rho-1)^{\rho-1}} \geq \limsup_{k \to \infty} \frac{\lambda_k}{(\log |a_k|^{-1/\lambda_k})^{\rho-1}}.$$

(3.2.4) and (3.2.5) obviously hold if $T = \infty$.

To prove the reverse inequality let lim sup in (3.2.1) be β. Since for $\beta = \infty$, this inequality would be obvious, we assume that $\beta < \infty$. For any $\varepsilon > 0$, we have for all $k > k_o = k_o(\varepsilon)$,

$$(3.2.6) \qquad |a_k| < \exp\{-\lambda_k \exp^{[q-2]} (\frac{\log^{[p-2]} \lambda_k}{\beta+\varepsilon})^{1/(\rho-A)}\}.$$

Now,

$$M(r) \leq \sum_{k=0}^{\infty} |a_k| \, r^{\lambda_k} = A(k_o) + \sum_{k=k_o+1}^{S} |a_k| \, r^{\lambda_k} + \sum_{k=S+1}^{\infty} |a_k| \, r^{\lambda_k},$$

where $A(k_o)$ is a polynomial of degree at most k_o and

$$S = [\exp^{[p-2]} \{(\beta+\varepsilon) \, (\log^{[q-1]} 2r)^{\rho-A}\}].$$

The last inequality in view of (3.2.6) gives,

$$(3.2.7) \quad M(r) < A(k_o)+S \max_{k \geq 0} [\exp\{-\lambda_k \exp^{[q-2]} (\frac{\log^{[p-2]} \lambda_k}{\beta+\varepsilon})^{1/(\rho-A)}\} r^{\lambda_k}] + \sum_{k=0}^{\infty} 2^{-k}.$$

Let,

$$\log H(r) = \max_{0 \leq x < \infty} \{P_x\},$$

where,

$$P_x = -x \exp^{[q-2]} (\frac{\log^{[p-2]} x}{\beta+\varepsilon})^{L^*} + x \log r \; ; \; L^* = \frac{1}{\rho-A}.$$

P_x assumes its maximum value at a point x_o satisfying

$$(3.2.8)\qquad \exp^{[q-2]}\left(\frac{\log^{[p-2]} x}{\beta+\varepsilon}\right)^{L^*} + \frac{x\, L^*\, E_{[q-2]}\left(\frac{\log^{[p-2]} x}{\beta+\varepsilon}\right)^{L^*}}{\Lambda_{[p-2]}(x)} = \log r.$$

x_o is uniquely defined by the above equation since $r \to \infty$ and we may confine to large enough values of r for which the left hand member of (3.2.8) is strictly increasing function of x.

By (3.2.8)

$$(3.2.9)\qquad P_{x_o} = \frac{x_o^2\, L^*\, E_{[q-2]}\left(\frac{\log^{[p-2]} x_o}{\beta+\varepsilon}\right)^{L^*}}{\Lambda_{[p-2]}(x_o)}.$$

For $(p,q) = (2,2)$, (3.2.8) gives $x_o = (\beta+\varepsilon)\{(\log r)/(1+L^*)\}^{1/L^*}$ and so

$$\log H(r) = \frac{(\beta+\varepsilon)\, L^*}{(1+L^*)\,(1+L^*)/L^*}(\log r)^{(1+L^*)/L^*}.$$

Using (3.2.6) and (3.2.7), for $(p,q) = (2,2)$ this ultimately leads to

$$(3.2.10)\qquad \limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^{\rho}} \leq \frac{(\rho-1)^{\rho-1}}{\rho^{\rho}}\,\beta .$$

For $(p,q) = (2,1)$, (3.2.8) gives $x_o = \{(\beta+\varepsilon)\, r^{1/L^*}\}$ and therefore $\log H(r) = \{(\beta+\varepsilon)\, L^*\, r^{1/L^*}\}/e$. Using (3.2.6) and (3.2.7) for $(p,q) = (2,1)$, we get,

$$(3.2.11)\qquad \limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho}} \leq \beta/e\rho .$$

Further, for all entire functions having index pair (p,q) such that $p \geq 3$, (3.2.8) gives $x_o = \exp^{[p-2]}\{(\beta+\varepsilon)(\log^{[q-1]} r)^{\rho}\}$ and so by (3.2.9) we have, for $p \geq 3$,

$$\log^{[2]} P_{x_o} \sim \log x_o \sim \exp^{[p-3]} \{(\beta+\varepsilon)\ (\log^{[q-1]} r)^{\rho}\} \text{ as } r \to \infty.$$

Now by (3.2.6) and (3.2.7), for sufficiently large values of r we have

$$\log^{[2]} M(r) < \log^{[2]} H(r)\ (1+o(1))$$

$$\sim \exp^{[p-3]} \{(\beta+\varepsilon)\ (\log^{[q-1]} r)^{\rho}\} \text{ as } r \to \infty.$$

Thus for entire functions having index pair (p,q) such that $p \geq 3$,

$$\limsup_{r \to \infty} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho}} \leq \beta. \tag{3.2.12}$$

Combining (3.2.10), (3.2.11) and (3.2.12), we get

$$\limsup_{r \to \infty} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho}} \leq M\ \beta, \tag{3.2.13}$$

(3.2.4), (3.2.5) and (3.2.13) together prove the theorem.

REMARK. (3.2.1) generalizes a result of Sato [68] which he obtained for entire functions with index pair (p,1). (3.2.1) also includes (1.4.9) for $\lambda_n = n$ and $1 < \alpha_1 < \infty$, $\rho = \alpha_2 = \ldots = \alpha_k = 0$.

3.3. In this section we obtain coefficient equivalents of the lower (p,q)-type of an entire function. Theorem 3.4 gives a formula for the lower (p,q)-type of an entire function when the coefficients of its power series (2.1.1) do not vary too rapidly. In Theorem 3.5 we find a coefficient formula for the lower (p,q)-type of the entire functions whose sequence of principal indices $\{\lambda_{n_k}\}_{k=0}^{\infty}$ is such that $\log^{[p-2]} \lambda_{n_{k-1}} \sim \log^{[p-2]} \lambda_{n_k}$ as $k \to \infty$.

First we observe that a coefficient formula analogous to (3.2.1) does not hold always for the lower (p,q)-type of an entire function. This can be seen by the example given in section 2.5. In that example, it is easily seen that if $q > 2$, then $T(p,q) = t(p,q) = 1$ while

$$\liminf_{k \to \infty} \frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho(p,q)}} = 0 .$$

However, the following relation always holds:

THEOREM 3.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function having* (p,q)-*order* ρ *and lower* (p,q)-*type* t *and let* $\{n_k\}_{k=0}^{\infty}$ *be an increasing sequence of positive integers. Then*

$$(3.3.1) \qquad t/M \geq G \liminf_{k \to \infty} \frac{\log^{[p-2]} \lambda_{n_{k-1}}}{(\log^{[q-1]} |a_{n_k}|^{-1/\lambda_{n_k}})^{\rho-A}},$$

where A and M are defined by (3.1.3) and (3.2.2) respectively and

$$(3.3.2) \qquad G \equiv G(\{n_k\}) = \liminf_{k \to \infty} \left(\frac{\lambda_{n_{k-1}}}{\lambda_{n_k}}\right)^{1/(\rho-1)} \quad \text{for} \quad (p,q) = (2,2)$$

$$= 1 \quad \text{for all other index pairs } (p,q) \, .$$

PROOF. Let,

$$\liminf_{k \to \infty} \frac{M \log^{[p-2]} \lambda_{n_{k-1}}}{\left(\log^{[q-1]} |a_{n_k}|^{-1/\lambda_{n_k}}\right)^{\rho-A}} = \beta^*.$$

Then, $0 \leq \beta^* \leq \infty$. First assume $0 < \beta^* < \infty$. Then for any ε satisfying $\beta^* > \varepsilon > 0$ and for all $k > k_o = k_o(\varepsilon)$,

$$(3.3.3) \qquad |a_{n_k}| > \exp \{-\lambda_{n_k} \exp^{[q-2]} \left(\frac{M \log^{[p-2]} \lambda_{n_{k-1}}}{\beta^*-\varepsilon}\right)^{1/(\rho-A)}\} \, .$$

Let, for $(p,q) \neq (2,2)$,

$$\log r_k = \{M^* + \exp^{[q-2]} \left(\frac{M \log^{[p-2]} \lambda_{n_{k-1}}}{\beta^*-\varepsilon}\right)^{1/\rho}\}, \; k = 2,3,\ldots,$$

where $M^* = 1/\rho$ if $(p,q) = (2,1)$ and $M^* = 1$ for all other index pairs (p,q). Then, for $r_k \leq r \leq r_{k+1}$ and $(p,q) \neq (2,2)$, (3.3.3) gives

$$\log M(r) > - \lambda_{n_k} \exp^{[q-2]} \left(\frac{M \log^{[p-2]} \lambda_{n_{k-1}}}{\beta^*-\varepsilon}\right)^{1/\rho} + \lambda_{n_k} \log r_k$$

$$= M^* \lambda_{n_k}$$

$$> M^* \exp^{[p-2]} \{\frac{\beta^*-\varepsilon}{M} (\overset{\log}{\cancel{\exp}}{}^{[q-1]} (\frac{r}{\exp M^*}))^{\rho}\}.$$

Hence, for $(p,q) \neq (2,2)$,

(3.3.4) $\qquad t \geq \beta^*$.

For $(p,q) = (2,2)$, let

(3.3.5) $$\log r_k = \left(\frac{\lambda_{n_{k-1}}}{\beta^*-\varepsilon}\right)^{1/(\rho-1)} \rho^{-1/(\rho-1)}, \qquad k = 2,3,\ldots\ldots .$$

Since in this case $\rho > 1$, therefore $\log r_k$ is an indefinitely increasing function of k. Let $r_k \leq r \leq r_{k+1}$. Then using (3.3.3) and proceeding as above, we have for $(p,q) = (2,2)$ and for sufficiently large r,

$$\frac{\log M(r)}{(\log r)^\rho} > \frac{\lambda_{n_k}\left(\frac{\lambda_{n_{k-1}}}{\beta^*-\varepsilon}\right)^{1/(\rho-1)} \rho^{-\rho/(\rho-1)}}{(\log r_{k+1})^\rho} .$$

Putting the value of $\log r_{k+1}$ from (3.3.5) and proceeding to limits, we get

(3.3.6) $\qquad t \geq G\, \beta^*$.

(3.3.4) and (3.3.6) hold trivially if $\beta^* = 0$ and if $\beta^* = \infty$ above arguments can be repeated with an arbitrarily large number in place of $(\beta^* - \varepsilon)$ to give $t = \infty$. Hence the proof of the theorem is complete.

COROLLARY 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function having (p,q)-order ρ and lower (p,q)-type t and let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of positive integers. Then,*

(3.3.7) $$t/M \geq G\,H \liminf_{k\to\infty} \frac{\log^{[p-2]} \lambda_{n_k}}{(\log^{[q-1]} |a_{n_k}|^{-1/\lambda_{n_k}})^{\rho-A}} ,$$

where A, M and G are as defined by (3.1.3), (3.2.2) and (3.3.2) respectively and

$$H \equiv H(\{n_k\}) = \liminf_{k\to\infty} \frac{\log^{[p-2]} \lambda_{n_{k-1}}}{\log^{[p-2]} \lambda_{n_k}} .$$

COROLLARY 2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function having (p,q)-order ρ and lower (p,q)-type t and let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of positive integers such that $\log^{[p-2]} \lambda_{n_{k-1}} \sim \log^{[p-2]} \lambda_{n_k}$ as $k \to \infty$, then

$$t/M \geq \liminf_{k\to\infty} \frac{\log^{[p-2]} \lambda_{n_k}}{(\log^{[q-1]} |a_{n_k}|^{-1/\lambda_{n_k}})^{\rho-A}} \tag{3.3.8}$$

where A and M are defined by (3.1.3) and (3.2.2) respectively.

COROLLARY 3. If

(i) $\log^{[p-2]} \lambda_{k-1} \sim \log^{[p-2]} \lambda_k$ as $k \to \infty$

(ii) $W = \lim_{k\to\infty} \dfrac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}}$ exists

where

$0 < \rho$, $W < \infty$, $p \geq q$, $p \geq 2$ and A is defined by (3.1.3), then

$f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ is an entire function of perfectly regular (p,q)-growth, having (p,q)-order ρ and (p,q)-type MW, where M is defined by (3.2.2).

REMARK. A result of Basinger [8] and (1.4.9) for $\lambda_n = n$, $1 < \alpha_1 < \infty$ and $\rho = \alpha_2 = \ldots = \alpha_k = 0$, follow as particular cases of Theorem 3.2 by taking $(p,q) = (2,1)$ and $(p,q) = (2,2)$ respectively in (3.3.7).

For the proof of our next theorem we need the following lemma.

LEMMA 3.1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having (p,q)-order ρ, (p,q)-type T and lower (p,q)-type t, then

$$(3.3.9) \qquad \begin{matrix} T \\ t \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \mu(r)}{(\log^{[q-1]} r)^{\rho}},$$

where $\mu(r)$ is the maximum term of $f(z)$ for $|z| = r$.

Proof of the lemma follows easily by using (1.6.2), hence we omit its proof.

THEOREM 3.3. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having (p,q)-order ρ and lower (p,q)-type t such that $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)}$ is a nondecreasing function of k for $k > k_0$, then

$$(3.3.10) \qquad t/M \leq \liminf_{k \to \infty} \frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}}$$

where A and M are defined by (3.1.3) and (3.2.2) respectively.

PROOF. First let $0 < t < \infty$. Then by (3.3.9) for any ε satisfying $t > \varepsilon > 0$ and for all $r > r_0 = r_0(\varepsilon)$, we get

REMARK . A result of Basinger [8] and (1.4.9) for $\lambda_n = n$, $1 < \alpha_1 < \infty$ and $\rho = \alpha_2 = \ldots = \alpha_k = 0$, follow as particular cases of Theorem 3.2 by taking $(p,q) = (2,1)$ and $(p,q) = (2,2)$ respectively in (3.3.7).

For the proof of our next theorem we need the following lemma.

LEMMA 3.1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function having* (p,q)*-order* ρ, (p,q)*-type* T *and lower* (p,q)*-type* t, *then*

$$(3.3.9) \qquad \begin{matrix} T \\ t \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \mu(r)}{(\log^{[q-1]} r)^{\rho}},$$

where $\mu(r)$ *is the maximum term of* $f(z)$ *for* $|z| = r$.

Proof of the lemma follows easily by using (1.6.2), hence we omit its proof.

THEOREM 3.3. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function having* (p,q)*-order* ρ *and lower* (p,q)*-type* t *such that* $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)}$ *is a nondecreasing function of* k *for* $k > k_0$, *then*

$$(3.3.10) \qquad t/M \leq \liminf_{k \to \infty} \frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho - A}}$$

where A *and* M *are defined by* (3.1.3) *and* (3.2.2) *respectively.*

PROOF. First let $0 < t < \infty$. Then by (3.3.9) for any ε satisfying $t > \varepsilon > 0$ and for all $r > r_0 = r_0(\varepsilon)$, we get

$$(3.3.11) \qquad \log \mu(r) > \exp^{[p-2]} \{(t-\varepsilon)\ (\log^{[q-1]} r)^{\rho}\}.$$

Let $a_{k_1} z^{\lambda_{k_1}}$ and $a_{k_2} z^{\lambda_{k_2}}$ $(k_1 > k_o,\ \psi(k_1-1) > r_o)$ be two consecutive maximum terms of $f(z)$. Then since $\psi(k)$ is a nondecreasing function of k for $k > k_o$, we have for $k_1 \leq k \leq k_2-1$,

$$(3.3.12) \qquad \psi(k_1) = \psi(k_1+1) = \ldots = \psi(k) = \ldots = \psi(k_2-1)$$

and

$$(3.3.13) \qquad |a_k|\ r^{\lambda_k} = |a_{k_2}|\ r^{\lambda_{k_2}} \quad \text{for } r = \psi(k).$$

(3.3.11), (3.3.12) and (3.3.13) give

$$(3.3.14) \quad \log |a_k| + \lambda_k \log \psi(k) = \log |a_{k_2}| + \lambda_{k_2} \log \psi(k_2-1)$$

$$> \exp^{[p-2]} \{(t-\varepsilon)(\log^{[q-1]} \psi(k))^{\rho}\}.$$

Let,

$$X = \frac{\log^{[p-2]}\lambda_{k-1}}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}}.$$

Then, using (3.3.14), we get

$$(3.3.15)$$

$$X > \frac{\log^{[p-2]} \lambda_{k-1}}{\exp[(\rho-A)\log^{[q-1]} \{-\frac{1}{\lambda_k} \exp^{[p-2]}\{(t-\varepsilon)(\log^{[q-1]}\psi(k))^{\rho}\} + \log \psi(k)\}]}.$$

We note that the minimum value of the function

$$S(r) = \frac{\log^{[p-2]} \lambda_{k-1}}{\exp[(\rho-A) \log^{[q-1]} \{-\frac{1}{\lambda_k} \exp^{[p-2]} \{(t-\varepsilon)(\log^{[q-1]}(r))^{\rho}\} + \log r\}]}$$

is attained at a point $r = r_k$, satisfying

$$\frac{E_{[p-2]} \{(t-\varepsilon)(\log^{[q-1]} r)^{\rho}\}}{\Lambda_{[q-1]}(r)} = \frac{\lambda_k}{\rho r} .$$

Hence it can be easily seen that

$$(3.3.16) \qquad S(\psi(k)) \geq S(r_k) \geq \begin{cases} e\rho(t-\varepsilon) & \text{for } (p,q) = (2,1) \\ \rho^{\rho}(t-\varepsilon)/(\rho-1)^{\rho-1} & \text{for } (p,q) = (2,2) \\ (t-\varepsilon)\ (1+o(1))^{-1} & \text{for all other index pairs } (p,q) \end{cases}$$

(3.3.15) and (3.3.16) give $\liminf_{k\to\infty} X \geq t/M$. This inequality is obvious if $t=0$. When $t = \infty$, above arguments with an arbitrarily large number in place of $(t - \varepsilon)$ lead to $\liminf_{k\to\infty} X = \infty$. This completes the proof of the theorem.

REMARK. Results of Basinger [8] and (1.4.9) for $\lambda_n = n$, $1 < \alpha_1 < \infty$, $\rho = \alpha_2 = \dots = \alpha_k = 0$, can be obtained as particular cases of Theorem 3.3, by taking $(p,q) = (2,1)$ and $(p,q) = (2,2)$ respectively in (3.3.10).

Combining Theorems 3.2 and 3.3 we get the following theorem which generalizes a result of Shah [69] and (1.4.9) for $\lambda_n = n$, $1 < \alpha_1 < \infty$ and $\rho = \alpha_2 = \dots = \alpha_k = 0$.

THEOREM 3.4. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function having* (p,q)*-order* ρ *and lower* (p,q)*-type* t *such that*

$$(3.3.17) \qquad \psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$$ *is a nondecreasing function of* k *for* $k > k_0$;

$$(3.3.18) \qquad \log^{[p-2]} \lambda_{k-1} \sim \log^{[p-2]} \lambda_k \text{ as } k \to \infty,$$

then

$$(3.3.19) \qquad t/M = \liminf_{k \to \infty} \frac{\log^{[p-2]} \lambda_{k-1}}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}}$$

where A *and* M *are defined by* (3.1.3) *and* (3.2.2) *respectively*.

REMARK. A sort of converse of Corollary 3 to Theorem 3.2 is also true as is evident by combining Theorems 3.1 and 3.4. Thus, if (3.3.17) and (3.3.18) hold and if $f(z)$ is of perfectly regular (p,q)-growth, i.e.,

$0 < T = t < \infty$, then $W \equiv \lim_{k \to \infty} \dfrac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]}|a_k|^{-1/\lambda_k})^{\rho-A}}$ exists.

Now we prove our main theorem of the section which is coefficient equivalent of the lower (p,q)-type t for all those entire functions for which the principal indices $\{\lambda_{n_k}\}_{k=0}^{\infty}$ satisfy the asymptotic relation

$\log^{[p-2]}\lambda_{n_{k-1}} \sim \log^{[p-2]}\lambda_{n_k}$ as $k \to \infty$.

<u>THEOREM</u> 3.5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function having (p,q)-order ρ and lower (p,q)-type t and let $\{\lambda_{n_k}\}_{k=0}^{\infty}$ be the sequence of principal indices of $f(z)$ such that $\log^{[p-2]}\lambda_{n_{k-1}} \sim \log^{[p-2]}\lambda_{n_k}$ as $k \to \infty$, then

$$(3.3.20)\quad t/M = \max_{\{m_k\}} \left[G(\{m_k\}) \liminf_{k\to\infty} \frac{\log^{[p-2]}\lambda_{m_{k-1}}}{(\log^{[q-1]}|a_{m_k}|^{-1/\lambda_{m_k}})^{\rho-A}}\right]$$

where A and M are defined by (3.1.3) and (3.2.2) and $G(\{m_k\})$ is defined as in (3.3.2). Maximum in (3.3.20) is taken over all increasing sequences $\{m_k\}_{k=0}^{\infty}$ of natural numbers.

<u>PROOF</u>. Consider the function $g(z) = \sum_{k=0}^{\infty} a_{n_k} z^{\lambda_{n_k}}$. It is easily seen that $g(z)$ is an entire function and that, for every finite z, $f(z)$ and $g(z)$ have same maximum term. Hence in view of Lemmas 2.1 and 3.1, $f(z)$ and $g(z)$ have same (p,q)-order and lower (p,q)-type. Thus $g(z)$ is of (p,q)-order ρ and lower (p,q)-type t. Since

$$\log^{[p-2]}\lambda_{n_{k-1}} \sim \log^{[p-2]}\lambda_{n_k} \text{ as } k \to \infty \text{ and } \psi(n_k) \equiv |a_{n_k}/a_{n_{k+1}}|^{1/(\lambda_{n_{k+1}}-\lambda_{n_k})}$$

forms a nondecreasing function of k for $k > k_o$, $g(z)$ satisfies the hypotheses of Theorem 3.4, therefore by (3.3.19),

$$(3.3.21)\quad t/M = G(\{n_k\}) \liminf_{k\to\infty} \frac{\log^{[p-2]}\lambda_{n_{k-1}}}{(\log^{[q-1]}|a_{n_k}|^{-1/\lambda_{n_k}})^{\rho-A}}.$$

Further, by Theorem 3.2,

$$(3.3.22)\quad t/M \geq \max_{\{m_k\}} \left[G(\{m_k\}) \liminf_{k\to\infty} \frac{\log^{[p-2]} \lambda_{m_{k-1}}}{(\log^{[q-1]} |a_{m_k}|^{-1/\lambda_{m_k}})^{\rho-A}}\right].$$

Comparing (3.3.21) and (3.3.22), we get (3.3.20). Hence the theorem.

3.4. In this section we find some relations involving (p,q)-type, lower (p,q)-type and ratio of the two consecutive coefficients of the entire series (2.1.1).

We first prove a lemma.

LEMMA 3.2. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of nonzero complex numbers and let $\{\lambda_k\}_{k=0}^{\infty}$ be an increasing sequence of positive integers such that for some integer $q \geq 1$, $|a_k|^{-1/\lambda_k} > \exp^{[q-2]}(1)$ for all $k > k_o$. Then, for a pair of integers (p,q), $p \geq q \geq 1, p \geq 2$,*

$$(3.4.1)\quad YR \leq \liminf_{k\to\infty} \frac{M \log^{[p-2]} \lambda_{k-1}}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}}$$

$$\leq \limsup_{k\to\infty} \frac{M \log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}} \leq XQ$$

where

$$(3.4.2)\qquad R \equiv R(p,q) = \liminf_{k\to\infty} \frac{\log^{[p-2]} \lambda_{k-1}}{(\log^{[q-1]} |a_{k-1}/a_k|^{1/(\lambda_k-\lambda_{k-1})})^{\rho-A}},$$

$$(3.4.3)\qquad Q \equiv Q(p,q) = \limsup_{k\to\infty} \frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_{k-1}/a_k|^{1/(\lambda_k-\lambda_{k-1})})^{\rho-A}},$$

A is defined by (3.1.3), ρ is such that $A < \rho < \infty$, M is defined by (3.2.2), $X = 1/\rho$ for $(p,q) = (2,1)$ or $(p,q) = (2,2)$ and $X = 1$ for all other pairs of integers (p,q), and

$$Y = \frac{\exp(\alpha - 1)}{\rho} \quad \text{for} \quad (p,q) = (2,1)$$

$$= \frac{\alpha^{1/(\rho-1)}}{\rho} \left(\frac{\rho-1}{\rho-\alpha}\right)^{\rho-1} \text{for } (p,q) = (2,2)$$

$$= 1 \text{ for all other pairs of integers } (p,q),$$

α being given by

$$\alpha = \liminf_{k \to \infty} (\lambda_{k-1}/\lambda_k).$$

PROOF. Let,

$$(3.4.4) \qquad d \equiv d(p,q) = \liminf_{k \to \infty} \frac{M \log^{[p-2]} \lambda_{k-1}}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}},$$

and

$$(3.4.5) \qquad D \equiv D(p,q) = \limsup_{k \to \infty} \frac{M \log^{[p-2]} \lambda_k}{(\log^{[q-1]} |a_k|^{-1/\lambda_k})^{\rho-A}}.$$

Then the inequality (3.4.1) is equivalent to

$$YR \le d \le D \le X Q .$$

The left hand inequality, viz., $Y R \le d$ is obvious if R is negative or zero. Hence we first assume that $0 < R < \infty$. For any ε such that $R > \varepsilon > 0$ from (3.4.2) we get

$$\log |a_{m-1}/a_m| < (\lambda_m - \lambda_{m-1}) \exp^{[q-2]} \left(\frac{\log^{[p-2]} \lambda_{m-1}}{R-\varepsilon}\right)^{1/(\rho-A)},$$

for all $m > N = N(\varepsilon)$. Writing the above inequality for $m = N + 1$, $N + 2, \ldots, k$ and adding all such inequalities, we get for all $k > N$

$$\log |a_k|^{-1} < \log |a_N|^{-1} + \sum_{m=N+1}^{k} (\lambda_m - \lambda_{m-1}) F_1(\lambda_{m-1})$$

where,

$$F_1(t) = \exp^{[q-2]} \{(\log^{[p-2]} t)/(R-\varepsilon)\}^{1/(\rho-A)} .$$

This implies,

$$(3.4.6) \quad \log |a_k|^{-1} < \log |a_N|^{-1} + \lambda_k F_1(\lambda_{k-1}) - I(\lambda_{k-1}) - \lambda_N F_1(\lambda_N),$$

where

$$I(\lambda_{k-1}) = \int_{\lambda_N}^{\lambda_{k-1}} n(t)\, d(F_1(t)) \; ; \; n(t) = \lambda_m \text{ for } \lambda_{m-1} < t \leq \lambda_m.$$

Now,

$$(3.4.7) \quad I(\lambda_{k-1}) = \frac{1}{\rho-A} \int_{\lambda_N}^{\lambda_{k-1}} n(t) \frac{E_{[q-2]}\left(\frac{\log^{[p-2]} t}{R-\varepsilon}\right)^{1/(\rho-A)}}{\prod_{m=0}^{p-2} \log^{[m]} t}\, dt$$

$$\geq \frac{1}{\rho-A} \int_{\lambda_N}^{\lambda_{k-1}} \frac{t\, E_{[q-2]}\left(\frac{\log^{[p-2]} t}{R-\varepsilon}\right)^{1/(\rho-A)}}{\prod_{m=0}^{p-2} \log^{[m]} t}\, dt .$$

For $(p,q) = (2,1)$, by (3.4.7)

$$I(\lambda_{k-1}) \geq \frac{1}{\rho} \int_{\lambda_N}^{\lambda_{k-1}} dt = \frac{1}{\rho} (\lambda_{k-1} - \lambda_N).$$

Hence, by (3.4.6), for all $k > N$ and for $(p,q) = (2,1)$

$$\log |a_k|^{-1} < O(1) + \frac{\lambda_k}{\rho} \log \frac{\lambda_{k-1}}{R-\varepsilon} - \frac{1}{\rho} (\lambda_{k-1} - \lambda_N),$$

which gives,

$$\log |a_k|^{-\rho/\lambda_k} < o(1) + \log \frac{\lambda_{k-1}}{R-\varepsilon} - (\alpha-\varepsilon) .$$

Therefore, for all $k > N$ and for $(p,q) = (2,1)$,

$$\frac{\lambda_{k-1}}{e\rho} |a_k|^{\rho/\lambda_k} > (R-\varepsilon) \frac{\exp(\alpha-\varepsilon-1)}{\rho} .$$

We thus have for $(p,q) = (2,1)$,

(3.4.8) $$d \geq \frac{\exp(\alpha-1)}{\rho} R.$$

For $(p,q) = (2,2)$, by (3.4.7),

$$I(\lambda_{k-1}) \geq \frac{1}{\rho-1} \left(\frac{1}{R-\varepsilon}\right)^{1/(\rho-1)} \int_{\lambda_N}^{\lambda_{k-1}} t^{1/(\rho-1)} dt$$

$$= \frac{1}{\rho} \left(\frac{1}{R-\varepsilon}\right)^{1/(\rho-1)} [\lambda_{k-1}^{\rho/(\rho-1)} - \lambda_N^{\rho/(\rho-1)}].$$

Hence, by (3.4.6), for all $k > N$ and for $(p,q) = (2,2)$

$$\log |a_k|^{-1/\lambda_k} < o(1) + \left(\frac{\lambda_{k-1}}{R-\varepsilon}\right)^{1/(\rho-1)} [1 - \frac{1}{\rho} \frac{\lambda_{k-1}}{\lambda_k}].$$

Therefore, for $(p,q) = (2,2)$

$$\liminf_{k \to \infty} \frac{(\rho-1)^{\rho-1}}{\rho^{\rho}} \frac{\lambda_{k-1}}{(\log |a_k|^{-1/\lambda_k})^{\rho-1}} \geq \frac{(\rho-1)^{\rho-1}}{\rho^{\rho}} R(1 - \frac{\alpha}{\rho})^{-(\rho-1)},$$

which implies that for $(p,q) = (2,2)$

$$(3.4.9) \qquad d \geq R \frac{\alpha^{1/(\rho-1)}}{\rho} (\frac{\rho-1}{\rho-\alpha})^{\rho-1}.$$

Finally, for the index pair (p,q) such that $p \geq 3$, (3.4.7) gives

$$I(\lambda_{k-1}) \geq \exp^{[q-2]} (\frac{\log^{[p-2]} \lambda_N}{R-\varepsilon})^{1/\rho} \int_{\lambda_N}^{\lambda_{k-1}} G(t)\, dt,$$

where

$$G(t) = \frac{E_{[q-3]} (\frac{\log^{[p-2]} t}{R-\varepsilon})^{1/\rho}}{\prod_{m=1}^{p-2} \log^{[m]} t}.$$

$G(t)$ is a decreasing function of t and tends to zero as $t \to \infty$. Therefore, for the index pair (p,q) such that $p \geq 3$,

$$I(\lambda_{k-1}) \geq \exp^{[q-2]} (\frac{\log^{[p-2]} \lambda_N}{R-\varepsilon})^{1/\rho} (\lambda_{k-1} - \lambda_N)\, G(\lambda_{k-1}).$$

Hence by (3.4.6), for all $k > N$ and for the index pair (p,q) such that $p \geq 3$,

$$\log |a_k|^{-1} < \lambda_k \exp^{[q-2]} (\frac{\log^{[p-2]} \lambda_{k-1}}{R-\varepsilon})^{1/\rho} (1-o(1)).$$

This, for the pair of integers (p,q) such that $p \geq 3$, gives

(3.4.10) $$d \geq R .$$

Combining (3.4.8), (3.4.9) and (3.4.10), we get

(3.4.11) $$d \geq Y R .$$

(3.4.11) is obvious when $R = 0$ and when $R = \infty$, then the usual arguments give $d = \infty$.

To prove the relation $D \leq X Q$ we note that Q can not be negative, for, this will contradict the hypothesis $|a_k|^{-1/\lambda_k} > \exp^{[q-2]}(1)$. Now we first assume that $0 \leq Q < \infty$. Then by (3.4.3), for any $\varepsilon > 0$ and for all $m > N = N(\varepsilon)$, we get

$$\log |a_{m-1}/a_m| > (\lambda_m - \lambda_{m-1}) \exp^{[q-2]} \left(\frac{\log^{[p-2]} \lambda_m}{Q+\varepsilon}\right)^{1/(\rho-A)} .$$

Writing the above inequality for $m = N + 1, N + 2, \ldots, k$ and adding all such inequalities, we have

$$\log |a_k|^{-1} > \log |a_N|^{-1} + \sum_{m=N+1}^{k} (\lambda_m - \lambda_{m-1}) F_2 (\lambda_m)$$

where, $F_2(t) = \exp^{[q-2]}\left(\frac{\log^{[p-2]} t}{Q + \varepsilon}\right)^{1/(\rho - A)}$. This implies that for all $k > N$,

(3.4.12) $$\log |a_k|^{-1} > \log |a_N|^{-1} + \lambda_k F_2(\lambda_k) - S(\lambda_k) - \lambda_N F_2(\lambda_{N+1})$$

where

$$S(\lambda_k) = \int_{\lambda_{N+1}}^{\lambda_k} n(t)\, d(F_2(t)) \; ; \; n(t) = \lambda_m \text{ for } \lambda_m < t \leq \lambda_{m+1} .$$

Now,

$$(3.4.13)\quad S(\lambda_k) = \frac{1}{\rho-A}\int_{\lambda_{N+1}}^{\lambda_k} n(t)\,\frac{E_{[q-2]}\left(\frac{\log^{[p-2]} t}{Q+\varepsilon}\right)^{1/(\rho-A)}}{\prod_{m=0}^{p-2}\log^{[m]} t}\,dt$$

$$\le \frac{1}{\rho-A}\int_{\lambda_{N+1}}^{\lambda_k} \frac{t\,E_{[q-2]}\left(\frac{\log^{[p-2]} t}{Q+\varepsilon}\right)^{1/(\rho-A)}}{\prod_{m=0}^{p-2}\log^{[m]} t}\,dt\,.$$

For $(p,q) = (2,1)$, by (3.4.13),

$$S(\lambda_k) \le \frac{1}{\rho}\int_{\lambda_{N+1}}^{\lambda_k} dt = \frac{1}{\rho}(\lambda_k - \lambda_{N+1}).$$

Hence by (3.4.12), for all $k > N$ and for $(p,q) = (2,1)$

$$\log|a_k|^{-1} > O(1) + \frac{\lambda_k}{\rho}\log(\lambda_k/(Q+\varepsilon)) - \frac{1}{\rho}(\lambda_k - \lambda_{N+1}),$$

which gives, for $(p,q) = (2,1)$

$$(3.4.14)\qquad \rho D \le Q\,.$$

For $(p,q) = (2,2)$,

$$S(\lambda_k) \le \frac{1}{\rho-1}\int_{\lambda_{N+1}}^{\lambda_k} \left(\frac{t}{Q+\varepsilon}\right)^{1/(\rho-A)} dt = \frac{1}{\rho(Q+\varepsilon)^{1/(\rho-1)}}\left[\lambda_k^{\rho/(\rho-1)} - \lambda_{N+1}^{\rho/(\rho-1)}\right].$$

Hence by (3.4.12) for $k > N$ and for $(p,q) = (2,2)$

$$\log|a_k|^{-1} > o(1) + \frac{\lambda_k^{\rho/(\rho-1)}}{(Q+\varepsilon)^{1/(\rho-1)}}\left(\frac{\rho-1}{\rho}\right),$$

which gives, for $(p,q) = (2,2)$

$$(3.4.15) \qquad \rho D \leq Q.$$

Finally for the pair of integers (p,q) such that $p \geq 3$

$$S(\lambda_k) \leq \frac{1}{\rho} \exp^{[q-2]} \left(\frac{\log^{[p-2]} \lambda_k}{Q+\varepsilon}\right)^{1/\rho} \left[\left(\int_{\lambda_{N+1}}^{[\sqrt{\lambda_k}]} + \int_{[\sqrt{\lambda_k}]}^{\lambda_k}\right) H(t)\, dt\right],$$

where

$$H(t) = \frac{E_{[q-3]} \left(\frac{\log^{[p-2]} t}{Q+\varepsilon}\right)^{1/\rho}}{\prod_{m=1}^{p-2} \log^{[m]} t}.$$

$H(t)$ is a decreasing function of t and tends to zero as $t \to \infty$. Hence

$$(3.4.16)\ S(\lambda_k) \leq \frac{1}{\rho} \exp^{[q-2]} \left(\frac{\log^{[p-2]} \lambda_k}{Q+\varepsilon}\right)^{1/\rho} [(\sqrt{\lambda_k}-\lambda_{N+1})H(\lambda_{N+1})-(\lambda_k-\sqrt{\lambda_k})H(\sqrt{\lambda_k})].$$

Now by (3.4.12) and (3.4.16), for all $k > N$ and for the pair of integers (p,q) such that $p \geq 3$, we get

$$\log |a_k|^{-1} > o(1) + \lambda_k \exp^{[q-2]} \left(\frac{\log^{[p-2]} \lambda_k}{Q+\varepsilon}\right)^{1/\rho} (1-o(1)),$$

and therefore for the pair of integers (p,q) such that $p \geq 3$

$$(3.4.17) \qquad D \leq Q$$

(3.4.14), (3.4.15) and (3.4.17) together give

$$(3.4.18) \qquad D \leq X\, Q.$$

The last inequality is obvious if $Q = \infty$. (3.4.11) and (3.4.18) together prove the lemma.

THEOREM 3.6. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having (p,q)-order ρ, (p,q)-type T and lower (p,q)-type t, then

(3.4.19) $$YR \leq t \leq T \leq X\,Q\,,$$

where R, Q, X and Y are defined as in Lemma 3.2.

PROOF. Since $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ is of (p,q)-order ρ, the relation (3.4.1) holds for its coefficients with ρ occuring there as (p,q)-order of $f(z)$. Now (3.4.19) follows, since by applying Theorems 3.1 and 3.2 it is evident that D can be replaced by (p,q)-type T and lower (p,q)-type t is greater than or equal to d, where d and D are defined by (3.4.4) and (3.4.5).

REMARKS (i) From (3.4.19) it follows that if $\log^{[p-2]} \bar{\lambda}_{k-1} \sim \log^{[p-2]} \bar{\lambda}_k$ as $k \to \infty$ and $R = Q$, then $f(z)$ is of perfectly regular (p,q)-growth. But converse of this need not be true as can be seen by the following example:

Let $f(z) = e^{z^2} + e^z$.

Here $(p,q) = (2,1)$, $\rho = \lambda = 2$, $T = t = 1$ but $R = 0$ and $Q = \infty$.

(ii) (3.4.19) generalizes a result of Juneja [33] which was obtained for $(p,q) = (2,1)$ and $\lambda_k = k$. For $(p,q) = (2,2)$ and $\lambda_k = k$, (3.4.19) includes a result of Awasthi [4].

LEMMA 3.3. Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of nonzero complex numbers and $\{\lambda_k\}_{k=0}^{\infty}$ be an increasing sequence of positive integers such that for $k > k_0$, $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ is a nondecreasing function tending to infinity with k, then

(3.4.20) $$D \leq X\,Q \leq \frac{XD}{M}$$

where X and Q are defined as in Lemma 3.2 and M and D are defined by (3.2.2) and (3.4.5) respectively.

PROOF. First let $D < \infty$. Then for given $\varepsilon > 0$ and for all $k > k_0 = k_0(\varepsilon)$

$$\log |a_k|^{-1} > \lambda_k \exp^{[q-2]} \left(\frac{M \log^{[p-2]} \lambda_k}{D+\varepsilon}\right)^{1/(\rho-A)} . \tag{3.4.21}$$

Since

$$\log |a_k|^{-1} = \log |a_N|^{-1} + \log |a_N/a_{N+1}| + \dots + \log |a_{k-1}/a_k|$$

$$\leq \log |a_N|^{-1} + (\lambda_k - \lambda_N) \log \psi(k-1),$$

using (3.4.21)

$$\lambda_k \exp^{[q-2]} \left(\frac{M \log^{[p-2]} \lambda_k}{D+\varepsilon}\right)^{1/(\rho-A)} < o(1) + (\lambda_k - \lambda_N) \log \psi(k-1),$$

and therefore,

$$\frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} \psi(k-1))^{\rho - A}} < \frac{D+\varepsilon}{M} .$$

Now, on proceeding to limits $Q \leq D/M$. When $D = \infty$, this inequality is obviously true. The other inequality of (3.4.20) is contained in (3.4.1). Hence the proof of the lemma is complete.

THEOREM 3.7. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having (p,q)-order ρ and (p,q)-type T such that $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)}$ is a nondecreasing function of k for $k > k_0$, then

$$T \leq X\, Q \leq \frac{XT}{M} \tag{3.4.21}$$

where X and Q are as defined in Lemma 3.2 and M is defined by (3.2.2).

PROOF. Since $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ is an entire function of (p,q)-order ρ and $\psi(k)$ forms a nondecreasing function of k,(3.4.20) holds for its coefficients a_n's and exponent λ_n's with ρ occuring in (3.4.20) as its (p,q)-order. Also, by Theorem 3.1, D = T, hence (3.4.21) follows.

REMARK. This theorem improves results of Juneja [33] and Awasthi [4] which were obtained for $\lambda_k = k$ and (p,q) = (2,1) and (p,q) = (2,2), respectively.

3.5. We now obtain some theorems involving the ratio of consecutive coefficients of the entire series (2.2.1) and its growth numbers as defined by (3.1.2). Theorem 3.8 improves a result of Juneja [31] .

THEOREM 3.8. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having (p,q)-order ρ, (p,q)-growth number μ and lower (p,q)-growth number δ and let $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ be an increasing function of k for $k > k_0$, then

$$(3.5.1) \qquad \mu = Q \quad \text{and} \quad \delta = R$$

where R and Q are defined by (3.4.2) and (3.4.3) respectively.

PROOF. First let $0 < \delta$, $\mu < \infty$. Then for any ε such that $\delta > \varepsilon > 0$ and for all $r > r_0 = r_0(\varepsilon)$, we have by (3.1.2),

$$(\delta-\varepsilon)(\log^{[q-1]} r)^{\rho-A} < \log^{[p-2]} \nu(r) < (\mu+\varepsilon)(\log^{[q-1]} r)^{\rho-A}.$$

Since $\psi(k)$ is a nondecreasing function of k for $k > k_o$,

$$\nu(r) = \lambda_k \quad \text{for} \quad \psi(k-1) \leq r < \psi(k) .$$

Hence,

(3.5.2) $$(\delta-\varepsilon)(\log^{[q-1]} r)^{\rho-A} < \log^{[p-2]} \lambda_k < (\mu+\varepsilon)(\log^{[q-1]} r)^{\rho-A}.$$

Since the inequality on right hand side of (3.5.2) holds for all $r \geq \psi(k-1)$, we get

$$\log^{[p-2]} \lambda_k < (\mu+\varepsilon) \ (\log^{[q-1]} \psi(k-1))^{\rho-A}.$$

The last inequality gives

(3.5.3) $$\mu \geq \limsup_{k \to \infty} \frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} \psi(k-1))^{\rho-A}} .$$

Similarly,

(3.5.4) $$\delta \leq \liminf_{k \to \infty} \frac{\log^{[p-2]} \lambda_{k-1}}{(\log^{[q-1]} \psi(k-1))^{\rho-A}} .$$

Next there exists a sequence $\{r_p\}$ tending to infinity such that

$$\log^{[p-2]} \nu(r_p) > (\mu-\varepsilon) \ (\log^{[q-1]} r_p)^{\rho-A} \text{ for } p = 1,2,\ldots .$$

Since $\psi(k)$ is an indefinitely increasing function and tends to infinity with k, for every r_p, we can find an integer k_p such that $\psi(k_p-1) \leq r_p < \psi(k_p)$ and so we have, $\nu(r_p) = \lambda_{k_p}$. Thus, we get

$$\log^{[p-2]} \lambda_{k_p} > (\mu-\varepsilon) \{\log^{[q-1]} \psi(k_p-1)\} \quad \text{for } p = 1,2,\ldots,$$

which implies,

$$(3.5.5) \qquad \mu \leq \limsup_{k \to \infty} \frac{\log^{[p-2]} \lambda_k}{(\log^{[q-1]} \psi(k-1))^{\rho-A}} .$$

Similarly, it can be shown that

$$(3.5.6) \qquad \delta \geq \liminf_{k \to \infty} \frac{\log^{[p-2]}\lambda_{k-1}}{(\log^{[q-1]} \psi(k-1))^{\rho-A}} .$$

(3.5.3), (3.5.4), (3.5.5) and (3.5.6) together give (3.5.1).

THEOREM 3.9. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be an entire function having (p,q)-order* ρ *, (p,q)-growth number* μ *and lower (p,q)-growth number* δ *, then*

$$(3.5.7) \qquad \delta \leq \mu \liminf_{k \to \infty} \frac{\log^{[p-2]}\lambda_k}{\log^{[p-2]}\lambda_{k+1}} .$$

PROOF. Let $\alpha^* = \liminf_{k \to \infty} \dfrac{\log^{[p-2]}\lambda_k}{\log^{[p-2]}\lambda_{k+1}}$. If $\beta^* > \alpha^*$, there exists a sequence $\{c(k)\}$ such that

$$\log^{[p-2]} \lambda_{c(k)} < \beta^* \log^{[p-2]} \lambda_{c(k)+1}.$$

Let r_t be a value of r at which the central index $\nu(r)$ of $f(z)$ jumps from a value less than or equal to $\lambda_{c(t)}$ to a value greater than or equal to $\lambda_{c(t)+1}$. Now, since

$$\log^{[p-2]}\nu(r_t-0) \leq \log^{[p-2]}\lambda_{c(t)} < \beta^* \log^{[p-2]}\lambda_{c(t)+1} \leq \beta^* \log^{[p-2]} \nu(r_t+0),$$

therefore,

$$\delta \leq \limsup_{t \to \infty} \frac{\log^{[p-2]} \nu(r_t-0)}{(\log^{[q-1]}r_t)^{\rho-A}} \leq \beta^* \limsup_{t \to \infty} \frac{\log^{[p-2]}\nu(r_t+0)}{(\log^{[q-1]}r_t)^{\rho-A}} \leq \mu\, \beta^*.$$

Since the above inequality holds for every $\beta^* > \alpha^*$, we have $\delta \leq \mu\, \alpha^*$ and the proof of the theorem is complete.

<u>REMARK</u>. If $\delta = \mu$ then $\lim\limits_{k \to \infty} \dfrac{\log^{[p-2]}\lambda_k}{\log^{[p-2]}\lambda_{k+1}} = 1$.

3.6. In this section we obtain a new characterization of the exponential function in terms of the central index. We thus have :

<u>THEOREM</u> 3.10. <u>Let</u> $f(z) = 1 + \sum\limits_{n=1}^{\infty} a_n z^n$ $(a_n \geq 0)$ <u>be an entire function and</u> $\nu(r)$ <u>be its central index for</u> $|z| = r$. <u>Then</u> $\nu(r) = [r]$, <u>if and only if</u>, $f(z) = \exp(z)$.

<u>PROOF</u>. 'If' part of the theorem is obvious. Let $\nu(r) = [r]$. Then $\nu(r) = n$ for $n \leq r < n + 1$.

By (1.6.1), we have

$$\log \mu(r) = \log \mu(n) + \int_n^r \frac{\nu(t)}{t}\, dt$$

$$= n \log r - n \log n + \log \mu(n-1) + \int_{n-1}^{n} \frac{\nu(t)}{t}\, dt$$

$$= n \log r - \log n - (n-1) \log(n-1) + \log \mu(n-2) + \int_{n-2}^{n-1} \frac{\nu(t)}{t}\, dt$$

$$= n \log r - \log n - \log (n-1) - (n-2) \log (n-2) + \log \mu(n-2) .$$

Continuing this process, we ultimately get

$$\log \mu(r) = n \log r - \log n - \log (n-1) - \dots - \log 2 + \log \mu(1).$$

Since $\mu(1) = 1$, we have $\mu(r) = \frac{r^n}{n!}$. But $\mu(r) = a_n r^n$ for $n \leq r < n + 1$, therefore $a_n = 1/n!$. Thus

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} = \exp (z).$$

This completes the proof of the theorem.

CHAPTER 4

APPROXIMATION OF AN ENTIRE FUNCTION

4.1. Let $f(x)$ be a real valued continuous function defined on the closed interval $[-1,1]$ and let

$$E_n(f) = \inf_{p \in \pi_n} ||f - p||$$

where the norm is the supremum norm on $[-1,1]$, denote the minimum error in Chebyshev approximation of $f(x)$ over the set π_n of real algebraic polynomials of degree atmost n.

Bernstein [9, p. 118] showed that

$$\lim_{n \to \infty} E_n^{1/n}(f) = 0$$

if and only if, $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$. Varga [92] further showed that if $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$, then the rate at which $E_n^{1/n}(f)$ tends to zero depends on its order $\rho(0 < \rho < \infty)$ as given by (1.8.3). Reddy [56] extended Varga's result to cover the case when $f(z)$ is of infinite or zero order.

In this chapter we use the results of Chapters 2 and 3 to find a more precise rate of decrease of $E_n^{1/n}(f)$ for a wider class of entire functions. Our results, given in the following sections, include the results of Reddy and Varga.

Throughout this chapter we shall stick to the notations and definitions introduced in Chapters 2 and 3.

4.2. We prove the following:

Theorem 4.1. *Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function $f(z)$ of (p,q)-order ρ, then*

$$(4.2.1) \qquad \rho = P(L_a)$$

where,

$$L_a = \limsup_{k \to \infty} \frac{\log^{[p-1]} k}{\log^{[q-1]}\left(\frac{1}{k} \log \frac{1}{E_k(f)}\right)} .$$

PROOF. Following Bernstein [46, p. 76], for each $k \geq 0$, we have

$$(4.2.2) \qquad E_k(f) \leq \frac{2B(r)}{r^k (r-1)} \qquad \text{for each } r > 1,$$

where $B(r) = \max_{z \in I_r} |f(z)|$ and I_r with $r > 1$ denotes the closed interior of the ellipse with foci at ± 1, half-major axis $\frac{r^2+1}{2r}$ and half-minor axis $\frac{r^2-1}{2r}$. The closed discs $D_1(r)$ and $D_2(r)$ bound the ellipse I_r in the sense that

$$D_1(r) \equiv \{z : |z| \leq \frac{r^2-1}{2r}\} \subset I_r \subset D_2(r) \equiv \{z : |z| \leq \frac{r^2+1}{2r}\} .$$

From this inclusion it follows that

$$(4.2.3) \qquad M\left(\frac{r^2-1}{2r}\right) \leq B(r) \leq M\left(\frac{r^2+1}{2r}\right) \quad \text{for all } r > 1,$$

and hence we have

(4.2.4) $$\rho = \limsup_{r \to \infty} \frac{\log^{[p]} B(r)}{\log^{[q]} r} .$$

Now (4.2.2) gives $E_k(f) \leq B(r)\, r^{-k}$ for every $r > 3$ and $k = 0,1,2,\ldots$. Thus, for every $\eta > 0$, we have

(4.2.5) $$\sum_{k=0}^{\infty} E_k(f)\, r^k \leq \sum_{k=0}^{\infty} \frac{B(r+\eta)}{(r+\eta)^k}\, r^k = B(r+\eta)\, \frac{(r+\eta)}{\eta} .$$

Further, for each $k \geq 0$, there exists a unique $p_k(z) \varepsilon \pi_k$ such that

$$||f - p_k|| = E_k(f) ,$$

and since $||p_{k+1} - p_k||$ is bounded above by $2\, E_k(f)$, we have [46, p. 42],

(4.2.6) $$|p_{k+1}(z) - p_k(z)| \leq 2E_k(f)\, r^{k+1} \text{ for } z \varepsilon I_r \text{ and } r > 1.$$

Therefore we can write

$$f(z) = p_0(z) + \sum_{k=0}^{\infty} (p_{k+1}(z) - p_k(z)),$$

where the series converges in every bounded domain of the complex plane. So (4.2.6) gives

$$|f(z)| \leq |p_0(z)| + 2 \sum_{k=0}^{\infty} E_k(f)\, r^{k+1} \quad \text{for } z \varepsilon I_r ,$$

and consequently, from the definition of $B(r)$

(4.2.7) $$B(r) \leq A_0 + 2r \sum_{k=0}^{\infty} E_k(f)\, r^k .$$

Now consider the entire function

$$H(z) = \sum_{k=0}^{\infty} E_k(f)\, z^k .$$

We have from (4.2.5) and (4.2.7)

$$(4.2.8) \qquad B(r) \leq C\, r\, H(r) \leq C\, \frac{r(r+\eta)\, B(r+\eta)}{\eta}$$

where C is some positive constant. From (4.2.4) and (4.2.8) it follows that

$$(4.2.9) \qquad \rho = \limsup_{r \to \infty} \frac{\log^{[p]} H(r)}{\log^{[q]} r} .$$

Now applying Theorem 2.1 with $\lambda_k = k$ to $H(z)$ we obtain (4.2.1). Hence the proof of the theorem is complete.

REMARK Theorem 4.1 includes a result of Varga [92] who obtained (4.2.1) for $(p,q) = (2,1)$ and results of Reddy [56] who obtained (4.2.1) for $(p,q) = (p,1)$ and $(p,q) = (2,2)$.

THEOREM 4.2. Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function of (p,q)-order ρ and let $\phi(k) \equiv E_k(f)/E_{k+1}(f)$ form a nondecreasing sequence of k for $k > k_o$. Then

$$(4.2.10) \qquad \rho = P(L_a^*)$$

where,

$$L_a^* = \limsup_{k \to \infty} \frac{\log^{[p-1]} k}{\log^{[q]}(E_{k-1}(f)/E_k(f))} .$$

PROOF. In view of (4.2.9), applying Theorem 2.2 with $\lambda_k = k$, to the entire function $H(z) = \sum_{k=0}^{\infty} E_k(f) z^k$, we get (4.2.10).

THEOREM 4.3. Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function $f(z)$ of (p,q)-order ρ. Let $\{n_k\}$ be the sequence of principal indices and $\rho(n_k)$ be the jump points of the central index of the function $H(z) = \sum_{k=0}^{\infty} E_k(f)\, z^k$. Then

$$(4.2.11) \qquad \rho = P\,(U_a)$$

where,

$$U_a = \lim_{k \to \infty} \sup \frac{\log^{[p-1]} n_k}{\log^{[q]} \rho(n_k)} .$$

PROOF. In view of (4.2.9), applying the corollary to Theorem 2.2 with $\lambda_k = k$, to the entire function $H(z)$, we get (4.2.11).

THEOREM 4.4. Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function of lower (p,q)-order λ and let $\phi(k) \equiv E_k(f)/E_{k+1}(f)$ form a nondecreasing function of k for $k > k_0$. Then

$$(4.2.12) \qquad \lambda = P(\omega_a)$$

$$(4.2.13) \qquad \quad = P(\omega_a^*)$$

where

$$\omega_a = \lim_{k \to \infty} \inf \frac{\log^{[p-1]} k}{\log^{[q-1]} \left(\frac{1}{k} \log \frac{1}{E_k(f)} \right)}$$

and

$$\omega_a^* = \lim_{k \to \infty} \inf \frac{\log^{[p-1]} k}{\log^{[q]} \left(E_{k-1}(f)/E_k(f)\right)}$$

PROOF. From (4.2.3) and (4.2.8), we have

$$(4.2.14) \qquad \lambda = \liminf_{r \to \infty} \frac{\log^{[p]} B(r)}{\log^{[q]} r} = \liminf_{r \to \infty} \frac{\log^{[p]} H(r)}{\log^{[q]} r} .$$

Now applying Theorems 2.5 and 2.6 to the entire function $H(z) = \sum_{k=0}^{\infty} E_k(f) z^k$, we get (4.2.12) and (4.2.13).

REMARK. (4.2.12) generalizes results of Reddy [92] who obtained it for the case $(p,q) = (p,1)$ and $(p,q) = (2,2)$.

THEOREM 4.5. *Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function of lower (p,q)-order λ. Let $\{n_k\}$ be the sequence of principal indices and $\rho(n_k)$ be the jump points of the central index of the function $H(z) = \sum_{k=0}^{\infty} E_k(f) z^k$. Then, for $(p,q) \neq (2,2)$*

$$(4.2.15) \qquad \lambda = P(V_a)$$

where,

$$V_a = \liminf_{k \to \infty} \frac{\log^{[p-1]} n_{k-1}}{\log^{[q]} \rho(n_k)} .$$

Further, if $\log n_{k-1} \sim \log n_k$ as $k \to \infty$, then (4.2.15) holds for $(p,q) = (2,2)$ also.

PROOF. In view of (4.2.14), applying corollary to Theorem 2.6 to the function $H(z)$, the theorem follows.

THEOREM 4.6. Let $f(x)$ be a real valued continuous function which is restriction to $[-1,1]$ of an entire function of lower (p,q)-order λ. Then, for $(p,q) \neq (2,2)$,

$$(4.2.16) \qquad \lambda = \max_{\{m_k\}} [P_\chi(\ell_a)]$$

$$(4.2.17) \qquad = \max_{\{m_k\}} [P_\chi(\ell_a^*)]$$

where,

$$\chi \equiv \chi(\{m_k\}) = \liminf_{k \to \infty} \frac{\log m_{k-1}}{\log m_k},$$

$$\ell_a \equiv \ell_a(\{m_k\}) = \liminf_{k \to \infty} \frac{\log^{[p-1]} m_{k-1}}{\log^{[q-1]}(\frac{1}{m_k} \log \frac{1}{E_{m_k}(f)})},$$

and

$$\ell_a^* \equiv \ell_a^*(\{m_k\}) = \liminf_{k \to \infty} \frac{\log^{[p-1]} m_{k-1}}{\log^{[q-1]} \{\frac{1}{m_k - m_{k-1}} \log(E_{m_{k-1}}(f)/E_{m_k}(f))\}}.$$

Further, if $\{n_k\}$ is the sequence of principal indices of the function $H(z) = \sum_{k=0}^{\infty} E_k(f) z^k$ such that $\log n_{k-1} \sim \log n_k$ as $k \to \infty$, then (4.2.16) and (4.2.17) hold for $(p,q) = (2,2)$ also. Maximum in (4.2.16) and (4.2.17) is taken over all increasing sequence $\{m_k\}$ of natural numbers.

PROOF. In view of (4.2.14) applying Theorem 2.7 with $\lambda_k = k$ to the entire function $H(z) = \sum_{k=0}^{\infty} E_k(f) z^k$, the theorem follows.

REMARK. (4.2.16) generalizes a result of J.P. Singh [73] who obtained it for $(p,q) = (2,1)$.

4.3. In this section we obtain relations involving the (p,q)-type, lower (p,q)-type and $E_k(f)$.

THEOREM 4.7. *Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function $f(z)$ of (p,q)-order ρ and (p,q)-type T. Then*

$$(4.3.1) \qquad T/M_a = \limsup_{k \to \infty} \frac{\log^{[p-2]} k}{\{\log^{[q-1]} (1/E_k(f))^{1/k}\}^{\rho - A}}$$

where

(4.3.2) $A = 1$ *for* $(p,q) = (2,2)$ *and* $A = 0$ *for all other index pairs* (p,q)

and

$$(4.3.3) \qquad M_a = \frac{(\rho-1)^{\rho-1}}{\rho^{\rho}} \quad \text{for } (p,q) = (2,2)$$

$$= 2^{\rho}/e\rho \quad \text{for } (p,q) = (2,1)$$

$$= 2^{\rho} \quad \text{for } (p,q) = (p,1),\ p \geq 3$$

$$= 1 \quad \text{for all other index pairs } (p,q).$$

PROOF. From (4.2.3) and (4.2.8) it follows that

$$(4.3.4) \qquad B_a T = \limsup_{r \to \infty} \frac{\log^{[p-1]} B(r)}{(\log^{[q-1]} r)^{\rho}} = \limsup_{r \to \infty} \frac{\log^{[p-1]} H(r)}{(\log^{[q-1]} r)^{\rho}}$$

where $H(z) = \sum_{k=0}^{\infty} E_k(f)z^k$ and $B_a = 2^{-\rho}$ for $(p,q) = (p,1)$ and $B_a = 1$ for all other index pairs (p,q). Now applying Theorem 3.1 with $\lambda_k = k$ to the function $H(z)$, we get (4.3.1).

THEOREM 4.8. *Let $f(x)$ be a real valued continuous function which is restriction to $[-1,1]$ of an entire function $f(z)$ of (p,q)-order ρ and lower (p,q)-type t. Let $\{m_k\}$ be an increasing sequence of positive integers, then*

$$(4.3.5)\quad t/M_a \geq G(\{m_k\}) \liminf_{k\to\infty} \frac{\log^{[p-2]} m_{k-1}}{\{\log^{[q-1]} (1/E_{m_k}(f))^{1/m_k}\}^{\rho-A}},$$

where A and M_a are defined by (4.3.2) and (4.3.3) respectively, and

$$(4.3.6)\quad G(\{m_k\}) = \liminf_{k\to\infty} \left(\frac{m_{k-1}}{m_k}\right)^{1/(\rho-1)} \quad \text{for } (p,q) = (2,2)$$

$$= 1 \quad \text{for all other index pairs } (p,q).$$

PROOF. From (4.2.3) and (4.2.8), it follows that

$$(4.3.7)\quad B_a t = \liminf_{r\to\infty} \frac{\log^{[p-1]} B(r)}{(\log^{[q-1]} r)^{\rho}} = \liminf_{r\to\infty} \frac{\log^{[p-1]} H(r)}{(\log^{[q-1]} r)^{\rho}},$$

where $H(z) = \sum_{k=0}^{\infty} E_k(f)\, z^k$ and $B_a = 2^{-\rho}$ for $(p,q) = (p,1)$ and $B_a = 1$ for all other index pairs (p,q). Now applying Theorem 3.2 to $H(z)$ we get (4.3.5).

THEOREM 4.9. Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function $f(z)$ of (p,q)-order ρ and lower (p,q)-type t. If $\phi(k) \equiv E_k(f)/E_{k+1}(f)$ forms a nondecreasing function of k for $k > k_0$, then

$$(4.3.8) \qquad t/M_a = \liminf_{k \to \infty} \frac{\log^{[p-2]} k}{\{\log^{[q-1]}(1/E_k(f))^{1/k}\}^{\rho - A}}$$

where A and M_a are defined by (4.3.2) and (4.3.3) respectively .

PROOF. Applying Theorem 3.3 to the function $H(z) = \sum_{k=0}^{\infty} E_k(f)\, z^k$, and using (4.3.7), we get

$$t/M_a \le \liminf_{k \to \infty} \frac{\log^{[p-2]} k}{\{\log^{[q-1]}(1/E_k(f))^{1/k}\}^{\rho - A}} .$$

This when combined with (4.3.5) gives (4.3.8).

REMARK. Theorems 4.7 to 4.9 generalize results of Reddy [92] who obtained them for $(p,q) = (p,1)$ and $(p,q) = (2,2)$.

THEOREM 4.10. Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function $f(z)$ of (p,q)-order ρ and lower (p,q)-type t. If $\{n_k\}$ are the principal indices of the function $H(z) = \sum_{k=0}^{\infty} E_k(f) z^k$ such that $\log^{[p-2]} n_{k-1} \sim \log^{[p-2]} n_k$ as $k \to \infty$, then

$$(4.3.9) \qquad t/M_a = \max_{\{m_k\}} \left[G(\{m_k\}) \liminf_{k \to \infty} \frac{\log^{[p-2]} m_{k-1}}{\{\log^{[q-1]}(1/E_{m_k}(f))^{1/m_k}\}^{\rho - A}} \right]$$

where A, M_a and G are defined by (4.3.2), (4.3.3) and (4.3.6) respectively and maximum in (4.3.9) is taken over all increasing sequences $\{m_k\}$ of natural numbers.

PROOF. Applying Theorem 3.4 to the function $H(z) = \sum_{k=0}^{\infty} E_{n_k}(f) z^{n_k}$ and using (4.3.7), we get

$$t/M_a = G(\{n_k\}) \liminf_{k\to\infty} \frac{\log^{[p-2]} n_{k-1}}{\{\log^{[q-1]}(1/E_{n_k}(f))^{1/n_k}\}^{\rho-A}} . \tag{4.3.10}$$

But, by (4.3.5), we get

$$t/M_a \geq \max_{\{m_k\}} [G(\{m_k\}) \liminf_{k\to\infty} \frac{\log^{[p-2]} m_{k-1}}{\{\log^{[q-1]}(1/E_{m_k}(f))^{1/m_k}\}^{\rho-A}} \tag{4.3.11}$$

Now by comparing (4.3.10) and (4.3.11), we get (4.3.9).

THEOREM 4.11. Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function of (p,q)-order ρ, (p,q)-type T and lower (p,q)-type t. Then

$$B_a^* R_a \leq t \leq T \leq B_a^* Q_a \tag{4.3.12}$$

where

$$\begin{matrix} Q_a \\ R_a \end{matrix} = \lim_{k\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-2]} k}{\{\log^{[q-1]}(E_{k-1}(f)/E_k(f))\}^{\rho-A}} , \tag{4.3.13}$$

A is defined by (4.3.2) and

(4.3.14) $B_a^* = 2^\rho/\rho$ for $(p,q) = (2,1)$, $B_a^* = 1/\rho$ for $(p,q) = (2,2)$, $B_a^* = 2^\rho$ for $(p,q) = (p,1)$, $p \geq 3$ and $B_a^* = 1$ for all other index pairs (p,q).

PROOF. Consider the function $H(z) = \sum_{k=0}^{\infty} E_k(f)z^k$. Then in view of (4.3.4), (4.3.7) and Theorem 3.6 we get (4.3.12).

THEOREM 4.12. Let $f(x)$ be a real valued continuous function which is the restriction to $[-1,1]$ of an entire function of (p,q)-order ρ and (p,q)-type T. If $\phi(k) \equiv E_k(f)/E_{k+1}(f)$ forms a nondecreasing function of k for $k > k_0$, then

$$(4.3.15) \qquad T \leq B_a^* \; Q_a \leq \frac{B_a^* \; T}{M_a},$$

where M_a, Q_a and B_a^* are defined by (4.3.3), (4.3.13) and (4.3.14) respectively.

PROOF. Let $H(z) = \sum_{k=0}^{\infty} E_k(f)z^k$. Then in view of (4.3.4) and Theorem 3.7, (4.3.15) follows.

CHAPTER 5

POLYNOMIAL COEFFICIENTS OF AN ENTIRE FUNCTION

5.1. Let $f(z)$ be an entire function. We have seen that growth of its maximum modulus $M(r)$, for $|z| = r$, is determined by its order ρ and type T where ρ and T are given by (1.2.2) and (1.2.3) respectively. Further, if $f(z)$ is represented by the power series $\sum_{n=0}^{\infty} a_n z^n$ then (1.3.2) and (1.3.3) respectively determine ρ and T in terms of the coefficients a_n.

Recently, Rice [57] has generalized these results in a different direction. He has considered the polynomial expansion of $f(z)$ of the form

$$(5.1.1) \qquad \sum_{k=1}^{\infty} q_k(z) \quad [p(z)]^{k-1}$$

where $p(z)$ is a polynomial of degree ζ and $q_k(z)$ is a uniquely determined polynomial of degree $\zeta-1$ or less. If Γ_R is the lemniscate $\Gamma_R = \{z: |p(z)| = R\}$ and $M(\Gamma_R) = \|f(z)\|_{\Gamma_R} = \max_{z \in \Gamma_R} |f(z)|$, then he has shown that $f(z)$ is an entire function of order ρ, if and only if,

$$(5.1.2) \qquad \limsup_{R \to \infty} \frac{\log \log M(\Gamma_R)}{\log R} = \rho/\zeta ;$$

Further, $f(z)$ is an entire function of order $\rho > 0$ and type $T(0 < T < \infty)$, if and only if,

$$(5.1.3) \qquad \limsup_{R \to \infty} \frac{\log M(\Gamma_R)}{R^{\rho/\zeta}} = T .$$

He has also generalized (1.3.2) and (1.3.3) by obtaining ρ and T in terms of the polynomial coefficients $q_k(z)$. Thus, it has been shown that if α be fixed, then $f(z)$ is an entire function of order $\rho > 0$, if and only if,

$$(5.1.4.) \qquad \rho = \zeta \limsup_{k \to \infty} \frac{k \log k}{\log ||q_k(z)||_{\Gamma_\alpha}^{-1}} .$$

Further, $f(z)$ is an entire function of order $\rho > 0$ and type T $(0 < T < \infty)$, if and only if,

$$(5.1.5) \qquad e \rho T = \zeta \limsup_{k \to \infty} k \left(||q_k(z)||_{\Gamma_\alpha}\right)^{\rho/\zeta n} .$$

Rice has also obtained an estimate for the length $||\Gamma_R||$ of the lemniscate Γ_R. Thus

$$(5.1.6) \qquad ||\Gamma_R|| = 2\pi R^{1/\zeta} (1 + o(1)) \quad \text{as} \quad R \to \infty .$$

His generalization of 'Cauchy's inequality' reads as follows. Let $f(z)$ be analytic in Γ_R, then there exists a polynomial $Q(z)$ of degree $\zeta-1$, independent of n and R, such that for $\alpha < R$

$$(5.1.7) \qquad ||q_k(z)||_{\Gamma_\alpha} \leq \frac{||\Gamma_R|| \, M(\Gamma_R)}{2\pi \, R^k} \, ||Q(z)||_{\Gamma_R} .$$

In this chapter, making use of the results obtained in Chapters 2 and 3, we generalize the results (5.1.2) to (5.1.5) by obtaining the corresponding results for (p,q)-order and (p,q)- type. Our methods also give a simple and short proof of (5.1.4) and (5.1.5). We also obtain lower (p,q)-order and

lower (p,q)-type in terms of the coefficient polynomials. These results generalize the corresponding results of Chapters 2 and 3. Throughout this chapter, we use the definitions and notations introduced in Chapters 2 and 3.

5.2. In this section we prove a few lemmas which will be required in the sequel. First we have

LEMMA 5.1. *f(z), given by (5.1.1), is an entire function of (p,q)-order ρ, if and only if,*

$$(5.2.1) \qquad \limsup_{R \to \infty} \frac{\log^{[p]} M(\Gamma_R)}{\log^{[q]} R^{1/\zeta}} = \rho$$

where ζ is the degree of p(z).

PROOF. From (5.1.6), $z \in \Gamma_R$ implies $|z| = R^{1/\zeta}(1 + o(1))$. Now setting $S = R^{1/\zeta}(1 + o(1))$, we have

$$\limsup_{R \to \infty} \frac{\log^{[p]} M(\Gamma_R)}{\log^{[q]} R^{1/\zeta}} = \limsup_{S \to \infty} \frac{\log^{[p]} M(|z| = S)}{\log^{[q]} \left(\frac{S}{1+o(1)} \right)} = \rho .$$

Hence the lemma.

Proceeding on the similar lines we have

LEMMA 5.2. *f(z), given by (5.1.1), is an entire function of lower (p,q)-order λ, if and only if,*

$$(5.2.2) \qquad \liminf_{R \to \infty} \frac{\log^{[p]} M(\Gamma_R)}{\log^{[q]} R^{1/\zeta}} = \lambda .$$

where ζ is the degree of $p(z)$.

LEMMA 5.3. Let $f(z)$, given by (5.1.1), be an entire function of (p,q)-order ρ. Then it is of (p,q)-type T, if and only if,

$$(5.2.3) \qquad \limsup_{R \to \infty} \frac{\log^{[p]} M(\Gamma_R)}{(\log^{[q-1]} R^{1/\zeta})^{\rho}} = T$$

where ζ is the degree of $p(z)$.

PROOF. Proceeding as in Lemma 5.1, we have

$$\limsup_{R \to \infty} \frac{\log^{[p]} M(\Gamma_R)}{(\log^{[q]} R^{1/\zeta})^{\rho}} = \limsup_{S \to \infty} \frac{\log^{[p-1]} M(|z| = S)}{\{\log^{[q-1]}(\frac{S}{1+o(1)})\}^{\rho}}$$

hence by the definition of (p,q)-type, (5.2.3) follows.

Similarly we have

LEMMA 5.4. Let $f(z)$, given by (5.1.1), be an entire function of (p,q)-order ρ. Then it is of lower (p,q)-type t, if and only if,

$$(5.2.4) \qquad \liminf_{R \to \infty} \frac{\log^{[p]} M(\Gamma_R)}{(\log^{[q-1]} R^{1/\zeta})^{\rho}} = t$$

where ζ is the degree of $p(z)$.

REMARK. Lemmas 5.1 and 5.3 generalize (5.1.2) and (5.1.3) respectively.

Now we prove our main lemma of this section which will be used as a basic tool in obtaining the subsequent theorems.

LEMMA 5.5. Let α be fixed† and $f(z) = \sum_{k=1}^{\infty} q_k(z)\,[p(z)]^{k-1}$ be an entire function having (p,q)-order ρ, lower (p,q)-order λ, (p,q)-type T

† Throughout our discussions in this chapter, α will denote a fixed constant not less than 1.

and lower (p,q)-type t, then

$$(5.2.5) \qquad \begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{R\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} H(R)}{\log^{[q]} R^{1/\zeta}}$$

and

$$(5.2.6) \qquad \begin{matrix}T \\ t\end{matrix} = \lim_{R\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} H(R)}{(\log^{[q-1]} R^{1/\zeta})^{\rho}}$$

where $H(R) = \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} R^k$ and ζ is the degree of $p(z)$.

PROOF. Let $R > \alpha$. Then, since [93, p. 77]

$$||q_k(z)||_{\Gamma_R} \leq ||q_k(z)||_{\Gamma_\alpha} R^{\zeta-1}$$

we have, for $z \in \Gamma_R$

$$|f(z)| \leq \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_R} ||p(z)||_{\Gamma_R}^{k-1}$$

and so

$$(5.2.7) \qquad M(\Gamma_R) \leq \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_R} ||p(z)||_{\Gamma_R}^{k-1}$$

$$\leq R^{\zeta-2} \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} R^k$$

$$= R^{\zeta-2} H(R).$$

Now using (5.1.6) and (5.1.7) it follows that for every $\eta > 0$

$$(5.2.8) \quad H(R) = \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} R^k$$

$$\leq M(\Gamma_{R+\eta}) \frac{||\Gamma_{R+\eta}|| \; ||Q||_{\Gamma_{R+\eta}}}{2\pi} \sum_{k=1}^{\infty} \left(\frac{R}{R+\eta}\right)^k$$

$$\leq M(\Gamma_{R+\eta}) \frac{(R+\eta)^{1/\zeta} (1+o(1)) \; ||Q||_{\Gamma_\alpha} R^{\zeta}}{2\pi \; \eta}$$

$$= M(\Gamma_{R+\eta}) \frac{R^{\zeta+\frac{1}{\zeta}} (1+o(1))}{2\pi \; \eta} ||Q||_{\Gamma_\alpha} .$$

Combining (5.2.7) and (5.2.8), we have for all $R > \alpha$ and $\eta > 0$,

$$(5.2.9) \qquad M(\Gamma_R) \leq R^{\zeta-2} H(R) \leq M(\Gamma_{R+\eta}) \frac{R^{2\zeta+\frac{1}{\zeta}-2} (1+o(1)) ||Q||_{\Gamma_\alpha}}{2\pi \; \eta} .$$

By using Lemmas 5.1 to 5.4, (5.2.5) and (5.2.6) follow from (5.2.9) after some simple calculations. Hence the lemma.

5.3. In this section using the results of Chapter 2, we obtain formulae for (p,q)-order and lower (p,q)-order involving the coefficient polynomials $q_k(z)$ of the entire function given by (5.1.1). Theorem 5.1 generalizes (5.1.4) and also gives a short method of obtaining it. Theorem 5.4 gives an analogous formula for the lower (p,q)-order λ which holds for a subclass of entire functions of the form (5.1.1). Theorem 5.6 gives lower (p,q)-order in terms of the coefficient polynomials $q_k(z)$. This formula holds for every entire function given by (5.1.1) which have index pair $(p,q) \neq (2,2)$. This result holds for $(p,q) = (2,2)$ also if the principal indices $\{n_k\}$ of the function $H(w) = \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} w^k$ satisfy the relation $\log n_{k-1} \sim \log n_k$ as $k \to \infty$.

The results contained in this section also generalize the corresponding results of Chapter 2.

THEOREM 5.1. *Let* α *be fixed and* $f(z) = \sum_{k=1}^{\infty} q_k(z)\,[p(z)]^{k-1}$ *be an entire function having* (p,q)*-order* ρ , *then*

$$(5.3.1) \qquad \rho = K\,P(L_d)$$

where

$$(5.3.2) \qquad L_d = \limsup_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q-1]}\{\frac{1}{k}\log \|q_k(z)\|_{\Gamma_\alpha}^{-1}\}}$$

and

(5.3.3) $\quad K = \zeta$ *if* $(p,q) = (p,1)$ *and* $K = 1$ *otherwise,* ζ *being the degree of* $p(z)$.

PROOF. Consider the function

$$H(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k .$$

It is easily seen that $H(w)$ is an entire function of index pair (p,q). Let its (p,q)-order be ρ^*. Then by Lemma 5.5, $\rho = K\rho^*$. Now applying Theorem 2.1 with $\lambda_k = k$ to $H(w)$ we get $\rho^* = P(L_d)$ and hence (5.3.1) follows.

THEOREM 5.2. *Let* α *be fixed and* $f(z) = \sum_{k=1}^{\infty} q_k(z)[p(z)]^{k-1}$ *be an entire function having* (p,q)*-order* ρ *such that* $\phi(k) \equiv (\|q_{k-1}(z)\|_{\Gamma_\alpha} / \|q_k(z)\|_{\Gamma_\alpha})$ *is a nondecreasing function of* k *for* $k > k_0$, *then*

(5.3.4) $$\rho = K\, P(L_d^*)$$

where

(5.3.5) $$L_d^* = \limsup_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q]} \{||q_{k-1}(z)||_{\Gamma_\alpha} / ||q_k(z)||_{\Gamma_\alpha}\}}$$

and K is defined by (5.3.3).

PROOF. Proceeding as in the proof of Theorem 5.1 we get $\rho = K\rho^*$. Now applying Theorem 2.2 with $\lambda_k = k$, to $H(w)$ we get $\rho^* = P(L_d^*)$. Hence (5.3.4) follows.

THEOREM 5.3. *Let* α *be fixed and* $f(z) = \sum_{k=1}^{\infty} q_k(z)[p(z)]^{k-1}$ *be an entire function of* (p,q)*-order* ρ. *Let* $\{n_k\}_{k=1}^{\infty}$ *be the sequence of principal indices and* $\rho(n_k)$ *be the jump points of the central index of the function* $H(w) = \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} w^k$. *Then,*

(5.3.6) $$\rho = K\, P(U_d)$$

where

$$U_d = \limsup_{k\to\infty} \frac{\log^{[p-1]} n_k}{\log^{[q]} \rho(n_k)} .$$

and K is defined by (5.3.3).

PROOF. In view of Lemma 5.5, applying the corollary to Theorem 2.2 to the entire function $H^*(w) = \sum_{k=1}^{\infty} ||q_{n_k}(z)||_{\Gamma_\alpha} w^{n_k}$, we get (5.3.6).

THEOREM 5.4. Let α be fixed and $f(z) = \sum_{k=1}^{\infty} q_k(z)\,[p(z)]^{k-1}$ be an entire function of lower (p,q)-order λ, such that $\phi(k)=(||q_{k-1}(z)||_{\Gamma_\alpha}/||q_k(z)||_{\Gamma_\alpha})$ forms a nondecreasing function of k for $k > k_o$. Then,

(5.3.7) $$\lambda = K\, P(\ell_d) = K\, P(\ell_d^*)$$

where

$$\ell_d = \liminf_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q-1]}\{\frac{1}{k}\log||q_k(z)||_{\Gamma_\alpha}^{-1}\}}$$

$$\ell_d^* = \liminf_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q]}\{||q_{k-1}(z)||_{\Gamma_\alpha}/||q_k(z)||_{\Gamma_\alpha}\}}$$

and K is defined by (5.3.3).

PROOF. Consider the entire function $H(w) = \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} w^k$, and let its lower order be denoted by λ^*. Then by Lemma 5.5, it follows that $\lambda = K\lambda^*$. Now applying Theorems 2.5 and 2.6 with $\lambda_k = k$ to $H(w)$, we get $\lambda^* = P(\ell_d) = P(\ell_d^*)$. Hence (5.3.7) follows.

THEOREM 5.5. Let α be fixed and $f(z) = \sum_{k=1}^{\infty} q_k(z)\,[p(z)]^{k-1}$ be an entire function of lower (p,q)-order λ. Let $\{n_k\}$ be the sequence of principal indices of the function $H(w) = \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} w^k$. Then, for $(p,q) \neq (2,2)$

(5.3.8) $$\lambda = K\, P(V_d)$$

where

$$V_d = \liminf_{k\to\infty} \frac{\log^{[p-1]} n_{k-1}}{\log^{[q]} \rho(n_k)} ,$$

and K is defined by (5.3.3). (5.3.8) holds for $(p,q) = (2,2)$ also if $\log n_{k-1} \sim \log n_k$ as $k \to \infty$.

PROOF. Using Lemma 5.5 and applying the corollary of Theorem 2.6 to the function $H(w)$, we get (5.3.8).

THEOREM 5.6. Let α be fixed and $f(z) = \sum_{k=1}^{\infty} q_k(z)\,[p(z)]^{k-1}$ be an entire function of lower (p,q)-order λ. Then, for $(p,q) \neq (2,2)$,

$$(5.3.9) \qquad \lambda/K = \max_{\{m_k\}} [P_\chi(\omega_d)] = \max_{\{m_k\}} [P_\chi(\omega_d^*)]$$

where,

$$\chi = \chi(\{m_k\}) = \liminf_{k\to\infty} \frac{\log m_{k-1}}{\log m_k} ,$$

$$\omega_d \equiv \omega_d(\{m_k\}) = \liminf_{k\to\infty} \frac{\log^{[p-1]} m_{k-1}}{\log^{[q-1]} \left(\frac{1}{m_k} \log \|q_{m_k}(z)\|_{\Gamma_\alpha}^{-1}\right)} ,$$

$$\omega_d^* = \omega_d^*(\{m_k\}) = \liminf_{k\to\infty} \frac{\log^{[p-1]} m_{k-1}}{\log^{[q-1]} \left\{\frac{1}{m_k - m_{k-1}} \log(\|q_{m_{k-1}}(z)\|_{\Gamma_\alpha} / \|q_{m_k}(z)\|_{\Gamma_\alpha})\right\}} ,$$

and K is defined by (5.3.3). Further if $\{n_k\}$ is the sequence of principal indices of the function $H(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k$ such that $\log n_{k-1} \sim \log n_k$ as $k \to \infty$, then (5.3.9) holds for $(p,q) = (2,2)$ also. Maximum in (5.3.9) is taken over all increasing sequences $\{m_k\}$ of natural numbers.

PROOF. Consider the entire function $H(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k$. Applying Lemma 5.5 to $H(w)$, we get $\lambda = K\lambda^*$, where λ^* is the lower-(p,q)-order of $H(w)$. Now applying Theorem 2.7 to $H(w)$ we get (5.3.9).

5.4. In this section using the results of Chapter 3, we obtain formulae for (p,q)-type and lower (p,q)-type involving the coefficient polynomials $q_k(z)$ of an entire function given by (5.1.1). Theorem 5.7 generalizes (5.1.5), whereas Theorem 5.8 gives a formula for the lower (p,q)-type involving the coefficient polynomials $q_k(z)$ which holds for a subclass of entire functions given by (5.1.1). Another polynomial coefficient formula for the lower (p,q)-type, which holds for a different subclass of entire functions given by (5.1.1), is obtained in Theorem 5.9. In the last two theorems of this section we obtain relations involving (p,q)-type, lower (p,q)-type and the ratio $(||q_{k-1}(z)||_{\Gamma_\alpha}/||q_k(z)||_{\Gamma_\alpha})$.

THEOREM 5.7. *Let* α *be fixed and* $f(z) = \sum_{k=1}^{\infty} q_k(z)\,[p(z)]^{k-1}$ *be an entire function of* (p,q)-*order* ρ *and* (p,q)-*type* T. *Then,*

$$T/M_d = \limsup_{k\to\infty} \frac{\log^{[p-2]} k}{\left(\log^{[q-1]} ||q_k(z)||_{\Gamma_\alpha}^{-1/k\zeta}\right)^{\rho-A}} \tag{5.4.1}$$

where

(5.4.2) $A = 1$ *for* $(p,q) = (2,2)$, $A = 0$ *for all other index pairs* (p,q),

(5.4.3) $M_d = \zeta/e\rho$ *for* $(p,q) = (2,1)$, $M_d = \zeta(\rho-1)^{\rho-1}/\rho^{\rho}$ *for* $(p,q) = (2,2)$ *and* $M_d = 1$ *for all other index pairs* (p,q), ζ *being the degree of* $p(z)$.

PROOF. Consider the function

$$H^*(w) = \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} w^{\zeta k} . \tag{5.4.4}$$

Then it is easily seen by Lemma 5.5 that $H^*(w)$ has (p,q)-order ρ and (p,q)-type T. Now applying Theorem 3.1 with $\lambda_k = k$ to $H^*(w)$, we get (5.4.1).

THEOREM 5.8. *Let α be fixed and $f(z) = \sum_{k=1}^{\infty} q_k(z)\,[p(z)]^{k-1}$ be an entire function of (p,q)-order ρ and lower (p,q)-type t such that $\phi(k) \equiv (\|q_{k-1}(z)\|_{\Gamma_\alpha}/\|q_k(z)\|_{\Gamma_\alpha})$ is a nondecreasing function of k for $k > k_0$, then*

$$(5.4.5) \qquad t/M_d = \liminf_{k \to \infty} \frac{\log^{[p-2]} k}{(\log^{[q-1]} \|q_k(z)\|_{\Gamma_\alpha}^{-1/k\zeta})^{\rho-A}},$$

where A and M_d are defined by (5.4.2) and (5.4.3) respectively and ζ is the degree of $p(z)$.

PROOF. Consider the function $H^*(w)$ defined by (5.4.4). Then it follows by Lemma 5.5 that $H^*(w)$ is of (p,q)-order ρ and lower (p,q)-type t. Applying Theorem 3.4 with $\lambda_k = k$, to the function $H^*(w)$, we get (5.4.5).

THEOREM 5.9. *Let α be fixed and $f(z) = \sum_{k=1}^{\infty} q_k(z)\ [p(z)]^{k-1}$ be an entire function of (p,q)-order ρ and lower (p,q)-type t. If $\{n_k\}$ are the principal indices of the function $H^*(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^{\zeta k}$ such that $\log^{[p-2]} n_{k-1} \sim \log^{[p-2]} n_k$ as $k \to \infty$, then*

$$(5.4.6) \quad t/M_d = \max_{\{m_k\}} \left[G(\{m_k\}) \liminf_{k \to \infty} \frac{\log^{[p-2]} m_{k-1}}{(\log^{[q-1]} \|q_{m_k}(z)\|_{\Gamma_\alpha}^{-1/\zeta m_k})^{\rho-A}}\right]$$

where A and M_d are defined by (5.4.2) and (5.4.3) respectively, ζ is the degree of $p(z)$ and

$$G(\{m_k\}) = \liminf_{k \to \infty} \left(\frac{m_{k-1}}{m_k} \right)^{1/(\rho-1)} \quad \text{for } (p,q) = (2,2)$$

$$= 1 \quad \text{for all other index pairs } (p,q).$$

PROOF. It follows by Lemma 5.5 that $H^*(w)$ is of (p,q)-order ρ and lower (p,q)-type t. Applying Theorem 3.5 to $H^*(w)$, we get (5.4.6).

THEOREM 5.10. Let α be fixed and $f(z) = \sum_{k=1}^{\infty} q_k(z)[p(z)]^{k-1}$ be an entire function of (p,q)-order ρ, (p,q)-type T and lower (p,q)-type t, then

(5.4.7) $$Y_d R_d \le t \le T \le Y_d Q_d$$

where

(5.4.8) $$\begin{matrix} Q_d \\ R_d \end{matrix} = \lim_{k \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-2]} k}{\{\log^{[q-1]} (||q_{k-1}(z)||_{\Gamma_\alpha}/||q_k(z)||_{\Gamma_\alpha})^{1/\zeta}\}^{\rho-A}},$$

(5.4.9) $Y_d = \frac{\zeta}{\rho}$ for $(p,q) = (2,1)$ or $(2,2)$ and $Y_d = 1$ for all other index pairs (p,q), ζ being the degree of $p(z)$.

PROOF. Consider the function $H^*(w)$ defined by (5.4.4). Then by Lemma 5.5 it follows that $H^*(w)$ has (p,q)-order ρ, (p,q)-type T and lower (p,q)-type t. Now applying Theorem 3.6 to $H^*(w)$, we get (5.4.7).

THEOREM 5.11. Let α be fixed and $f(z) = \sum_{k=1}^{\infty} q_k(z)\,[p(z)]^{k-1}$ be an entire function of (p,q)-order ρ, and (p,q)-type T such that $\phi(k) \equiv (||q_{k-1}(z)||_{\Gamma_\alpha} / ||q_k(z)||_{\Gamma_\alpha})$ is a nondecreasing function of k for $k > k_o$, then

$$(5.4.10) \qquad T/Y_d \leq Q_d \leq T/M_d$$

where M_d, Q_d and Y_d are defined by (5.4.3), (5.4.8) and (5.4.9) respectively.

PROOF. Consider the function $H^*(w)$ defined by (5.4.4), which by Lemma 5.5 is of (p,q)-order ρ and (p,q)-type T. Now applying Theorem 3.7 to $H^*(w)$, we get (5.4.10).

REMARK. From Theorem 5.11 it follows that if $\phi(k)$ is a nondecreasing function of k for $k > k_o$, then $T = Q_d$ for all index pairs (p,q) other than $(2,1)$ and $(2,2)$.

CHAPTER 6

GROWTH OF FUNCTIONS ANALYTIC IN A DISC

6.1. Let U denote the class of nonconstant functions analytic in the unit disc $D = \{z: |z| < 1\}$ and let the Taylor series expansion of a function $f(z) \in U$ be given by

$$(6.1.1) \qquad f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$$

where $\lambda_o = 0$ and $\{\lambda_k\}_{k=1}^{\infty}$ is the increasing sequence of those positive integers for which no element of the sequence $\{a_k\}_{k=1}^{\infty}$ is zero.

For $0 < r < 1$, set

$$M(r) \equiv M(r,f) = \max_{|z|=r} |f(z)|, \quad \mu(r) \equiv \mu(r,f) = \max_{n \geq 0} \{|a_n| r^{\lambda_n}\}$$

and

$$\nu(r) \equiv \nu(r,f) = \max \{\lambda_n : \mu(r) = |a_n| r^{\lambda_n}\} \quad ,$$

The function $\nu(r)$ is a nondecreasing function of r and has only ordinary discontinuities in the open interval $(0,1)$. The points of discontinuties of $\nu(r)$ are called its jump points and elements in the range set of $\nu(r)$ are called principal indices of $f(z)$.

The order ρ_o and lower order ζ_o of a function $f(z)$ of the class U are defined as

$$(6.1.2) \qquad \begin{matrix}\rho_o \\ \zeta_o\end{matrix} = \lim_{r \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^+ \log^+ M(r)}{-\log (1-r)}, \quad (0 \leq \zeta_o \leq \rho_o \leq \infty).$$

Following the lines of MacLane [49, p. 47], it can be easily shown that

$$(6.1.3)\qquad \frac{\rho_o}{1+\rho_o} = \limsup_{k\to\infty} \frac{\log^+\log^+|a_k|}{\log\lambda_k}.$$

Let A be the MacLane class of those functions of the class U which have asymptotic values at a dense subset of points of $|z| = 1$.

In this chapter we first obtain a sufficient condition on the coefficients a_k, so that a function of the class U, given by (6.1.1), belongs to the class A. This improves a result of MacLane [49, p. 51] and also shows that all those functions in U, which are of finite order, belong to A. Then, for functions belonging to a subclass of U, we obtain the order ρ_o in terms of the ratio of their consecutive coefficients. This result is used in finding a sufficient condition on $r_k (> 0)$ such that the function $f(z) = 1 + \sum_{k=1}^{\infty} (r_1 r_2 \dots r_k) z^{\lambda_k}$ belongs to the class A. Further, in section 6.3, we give a complete coefficient characterization for the lower order ζ_o of $f(z) \in U$, from which a result of L.R. Sons [75, Theorem 1] can be derived. We also give complete coefficient characterization of the lower order ζ_o of $f(z) \in U$ in terms of the ratio of the coefficients in (6.1.1). In the last section we prove the existence of a function $f(z)$ analytic in $D_R \equiv \{z: |z| < R\}$, $0 < R < \infty$ which possesses prescribed rates of growth on two different, although unspecified, sequences tending to R. Existence of a similar function for $D_R^c \equiv \{z: |z| > R\}$ is also proved.

6.2. In this section we prove two theorems. The first theorem gives a sufficient condition on the coefficients a_k, so that the function, given by (6.1.1), belongs to the class A. This weakens a similar condition of MacLane. In the second theorem we obtain the order of a function belonging to a subclass of U, in terms of the ratio of the two consecutive coefficients in its Taylor series expansion (6.1.1). This result is used in obtaining a sufficient condition on $r_k(>0)$, so that a function of the form $f(z) = 1 + \sum_{k=1}^{\infty} (r_1 r_2 \ldots r_k) z^{\lambda_k}$ belongs to the class A.

THEOREM 6.1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$, $|z| < 1$, *be a nonconstant function such that for some* $\beta (1 < \beta < \infty)$,

$$\log^+ |a_k| = 0 \{\lambda_k (\log \lambda_k)^{-1} (\log \log \lambda_k)^{-\beta}\}, \text{ as } k \to \infty, \tag{6.2.1}$$

then $f(z) \in A$.

We require a lemma.

LEMMA 6.1. *Let* $f(z) \in U$ *and suppose for some* $\alpha (1 < \alpha < \infty)$, *the maximum modulus* $M(r)$ *of* $f(z)$ *on* $|z| = r$ $(0 < r < 1)$ *satisfies*

$$\log^+ \log^+ M(r) = 0 \{(1-r)^{-1} (\log (e/(1-r)))^{-\alpha}\}, \text{ as } r \to 1, \tag{6.2.2}$$

then $f(z) \in A$.

PROOF. The lemma immediately follows in view of the fact that (6.2.2) implies

$$\int_0^1 \log^+ \log^+ M(r) < \infty,$$

and so, using Hornblower's result (1.9.11), we have $f(z) \in A$.

PROOF OF THEOREM 6.1. It is evident from (6.2.1) that $f(z) \in U$. Again, by (6.2.1), for all $k > k_o$, we get

$$\log^+ |a_k| < B\ \lambda_k (\log \lambda_k)^{-1} (\log\log \lambda_k)^{-\beta}$$

where B is a positive finite constant.

Let us write,

$$M(r) \leq \sum_{k=0}^{\infty} |a_k|\ r^{\lambda_k}$$

$$(6.2.3) \qquad = \sum_{k=0}^{k_o} |a_k|\ r^{\lambda_k} + \sum_{k=k_o+1}^{N} |a_k|\ r^{\lambda_k} + \sum_{k=N+1}^{\infty} |a_k|\ r^{\lambda_k},$$

where,

$$N = [\exp^{[2]} \{(\frac{1}{2B} \log \frac{1}{r})^{-1/(\beta+1)}\}].$$

Now,

$$(6.2.4) \qquad \sum_{k=N+1}^{\infty} |a_k|\ r^{\lambda_k} < \sum_{k=N+1}^{\infty} \exp\{B\ \lambda_k(\log \lambda_k)^{-1}(\log\log \lambda_k)^{-\beta}\}\ r^{\lambda_k}$$

$$\leq \sum_{k=N+1}^{\infty} r^{\lambda_k/2} \leq \frac{r^{(N+1)/2}}{1-r^{1/2}}.$$

In view of

$$(1 - r) = (\log \frac{1}{r})\ (1 + O\ (\log \frac{1}{r})), \text{ as } r \to 1,$$

we have

$$\log \frac{r^{(N+1)/2}}{1-r^{1/2}} \leq -\frac{1}{2}(\log \frac{1}{r})\exp^{[2]}\{(\frac{1}{2B} \log \frac{1}{r})^{-1/(\beta+1)}\} - \log (1 - r^{1/2})$$

$$= -\frac{1}{2}(\log \frac{1}{r})\exp^{[2]} \{(\frac{1}{2B} \log \frac{1}{r})^{-1/(\beta+1)}\} (1+o(1)), \text{ as } r \to 1.$$

Hence, by (6.2.4),

$$\sum_{k=N+1}^{\infty} |a_k| r^{\lambda_k} \leq \frac{r^{(N+1)/2}}{1-r^{1/2}} = o(1), \text{ as } r \to 1.$$

Thus, by (6.2.3)

$$(6.2.5)\quad M(r) < C(k_o) + N \max_{k \geq 0} [\exp\{B\lambda_k(\log \lambda_k)^{-1}(\log \log \lambda_k)^{-\beta}\} r^{\lambda_k}] + o(1),$$

where $C(k_o)$ is a finite constant depending on k_o.

Let,

$$g(x,r) = Bx(\log x)^{-1} (\log \log x)^{-\beta} + x \log r.$$

Maximum value of $g(x,r)$ occurs at a point $x = x_o \equiv x_o(r)$, satisfying the equation,

$$B(\log x)^{-1}(\log \log x)^{-\beta} \{1-(\log x)^{-1} - B\beta(\log x)^{-1}(\log \log x)^{-1}\} = \log \frac{1}{r}.$$

It is easily seen that $x_o(r) \to \infty$ as $r \to 1$. Hence,

$$x_o = \exp \{(1 + o(1)) B(\log \frac{1}{r})^{-1} (-\log \log \frac{1}{r})^{-\beta}\} .$$

Therefore

$$g(x,r) \leq B x_o (\log x_o)^{-1} (\log \log x_o)^{-\beta} \{1 + B \beta(\log \log x_o)^{-1}\}$$

and so,

$$\log g(x,r) \leq \log x_0 + o(1)$$

$$= B\,(1 + o(1))\,(\log \tfrac{1}{r})^{-1}\,(-\log\log \tfrac{1}{r})^{-\beta} + o(1).$$

Hence by (6.2.5), as $r \to 1$,

$$\log^+ M(r) < \log N + \exp\{B(1+o(1))(\log \tfrac{1}{r})^{-1}(-\log\log \tfrac{1}{r})^{-\beta} + o(1)\} + o(1)$$

$$\leq \exp\{(\tfrac{1}{2B}\log \tfrac{1}{r})^{-1/(\beta+1)}\} + \exp\{B(1+o(1))(\log\tfrac{1}{r})^{-1}(-\log\log\tfrac{1}{r})^{-\beta} + o(1)\} + o(1)$$

$$= \exp\{B(1+o(1))(\log \tfrac{1}{r})^{-1}(-\log\log \tfrac{1}{r})^{-\beta} + o(1)\}\,(1 + o(1))).$$

Since the right hand side of the above inequality is a positive quantity, therefore as $r \to 1$,

$$\log^+\log^+ M(r) \leq B(1+o(1))\,(\log \tfrac{1}{r})^{-1}\,(-\log\log \tfrac{1}{r})^{-\beta} + o(1)$$

$$= O\,\{(\log \tfrac{1}{r})^{-1}\,(-\log\log \tfrac{1}{r})^{-\beta}\}$$

$$= O\,\{(1-r)^{-1}\,(\log (e/(1-r)))^{-\beta}\}.$$

So, by Lemma 6.1, $f(z) \in A$. Hence the theorem.

<u>COROLLARY</u>. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ <u>be a function of the class</u> U <u>having finite order</u> ρ_0, <u>then</u> $f(z) \in A$.

Corollary follows immediately from Theorem 6.1 since if the function $f(z)$, belonging to U, is of finite order, then by (6.1.3), as $k \to \infty$.

$$\log^+ |a_k| = O(\lambda_k^p)\ ,\ 0 < p < 1,$$

$$= O(\lambda_k\,(\log \lambda_k)^{-1}\,(\log\log \lambda_k)^{-\beta}) \text{ for any } \beta > 1.$$

REMARKS. (i) Theorem 6.1 improves a result of MacLane [49, p. 51] who showed that functions of the class U having order less than 2 belong to class A and consequently, if the function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $|z| < 1$ is such that (1.9.10) is satisfied then $f(z) \varepsilon A$. Our condition (6.2.1) weakens MacLane's condition (1.9.10) considerably and shows that all those functions of class U which are of finite order belong to class A.

(ii) There are functions of infinite order also which belong to class A. Consider, for example,

$$f(z) = \sum_{k=0}^{\infty} \exp(\lambda_k (\log \lambda_k)^{-3}) z^{\lambda_k}.$$

Then from Theorem 6.1, it is clear that $f(z) \varepsilon A$ and from (6.1.3) it follows that the order of $f(z)$ is infinite.

THEOREM 6.2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *belong to the class* U *and have order* $\rho_o (0 \le \rho_o \le \infty)$. *If* $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)} \ge 1/e$ *for all* $k > k_o$, *then*

$$(6.2.6) \qquad 1 + \rho_o \le \max (1, \theta),$$

where,

$$(6.2.7) \qquad \theta = \limsup_{k \to \infty} \frac{\log \lambda_k}{\log \left(\frac{\lambda_k - \lambda_{k-1}}{\log^+ |a_k/a_{k-1}|} \right)}$$

quotient occuring at the right hand side of (6.2.7) being interpreted to be zero for the values of k for which $|a_k/a_{k-1}| \leq 1$. Further if

(6.2.8) $\psi(k)$ is a nondecreasing function of k for $k > k_\sigma$, then equality holds in (6.2.6).

PROOF. Since $\psi(k) \geq 1/e$ for all $k > k_1$, we have $0 \leq \theta \leq \infty$. First let $\theta < \infty$. For any β such that $\theta < \beta < \infty$, we get, by (6.2.7), for all $k \geq N = N(\beta)$,

$$\log^+ |a_k/a_{k-1}| < (\lambda_k - \lambda_{k-1})\, \lambda_k^{-1/\beta}.$$

Therefore, if $k > \max(N, k_1)$, then

$$\begin{aligned}
(6.2.9) \qquad \log |a_k| &= \log |a_N| + \log |a_{N+1}/a_N| + \ldots + \log |a_k/a_{k-1}| \\
&< \log |a_N| + (\lambda_{N+1} - \lambda_N)\, \lambda_{N+1}^{-1/\beta} + \ldots + (\lambda_k - \lambda_{k-1})\, \lambda_k^{-1/\beta} \\
&= \log |a_N| + \lambda_k^{(\beta-1)/\beta} - \sum_{m=N+1}^{k-1} \lambda_m (\lambda_{m+1}^{-1/\beta} - \lambda_m^{-1/\beta}) - \lambda_N\, \lambda_{N+1}^{-1/\beta} \\
&= \log |a_N| + \lambda_k^{(\beta-1)/\beta} - \int_{\lambda_{N+1}}^{\lambda_k} n(t)\, d(t^{-1/\beta}) - \lambda_N\, \lambda_{N+1}^{-1/\beta} ,
\end{aligned}$$

where $n(t) = \lambda_m$ for $\lambda_m < t \leq \lambda_{m+1}$; $m = N+1, \ldots, k-1$. Since

$$\begin{aligned}
\int_{\lambda_{N+1}}^{\lambda_k} n(t)\, d(t^{-1/\beta}) &= -\frac{1}{\beta} \int_{\lambda_{N+1}}^{\lambda_k} \frac{n(t)}{t}\, t^{-1/\beta}\, dt \\
&> -\frac{1}{\beta} \int_{\lambda_{N+1}}^{\lambda_k} t^{-1/\beta}\, dt \\
&= -\frac{1}{\beta-1} \{\lambda_k^{(\beta-1)/\beta} - \lambda_{N+1}^{(\beta-1)/\beta}\},
\end{aligned}$$

therefore, by (6.2.9),

$$(6.2.10)\quad \log |a_k| < \log |a_N| + \frac{\beta}{\beta-1}\lambda_k^{(\beta-1)/\beta} - \frac{1}{\beta-1}\lambda_{N+1}^{(\beta-1)/\beta} - \lambda_N \lambda_{N+1}^{-1/\beta} .$$

By (6.1.3), if $\theta < \beta < 1$, then (6.2.10) gives $\rho_o = 0$ and so (6.2.6) obviously holds. Hence suppose that $1 \leq \theta < \beta < \infty$. (6.2.10) then gives

$$\log^+ \log^+ |a_k| < \frac{(\beta-1)}{\beta} \log \lambda_k + o(1).$$

Using (6.1.3), the last inequality gives $\frac{\rho_o}{1+\rho_o} \leq \frac{\beta-1}{\beta}$. Since this holds for every $\beta > \theta$, we have $\frac{\rho_o}{1+\rho_o} \leq \frac{\theta-1}{\theta}$, which implies $1 + \rho_o \leq \theta$. If $\theta = \infty$, this inequality is trivially true. Hence the proof of (6.2.6) is complete.

Next if $0 \leq \theta \leq 1$, then $1 + \rho_o \geq \max(1, \theta)$ is trivially true. Hence, let us first assume that $1 < \theta < \infty$. Since $\psi(k)$ is a non-decreasing function of k for $k > k_o$, therefore for a fixed $N > k_o$ and for all k such that $k > N$, we have

$$\begin{aligned}\log |a_k| &= \log |a_N| + \log |a_{N+1}/a_N| + \dots + \log |a_k/a_{k-1}| \\ &= \log |a_N| + (\lambda_{N+1}-\lambda_N) \log \frac{1}{\psi(N)} + \dots + (\lambda_k - \lambda_{k-1})\log \frac{1}{\psi(k-1)} \\ &> \log |a_N| + (\lambda_k - \lambda_N) \log \frac{1}{\psi(k-1)} ,\end{aligned}$$

which gives for all $k > N$,

$$\log^+ \log^+ |a_k| > O(1) + \log (\lambda_k - \lambda_N) + \log \log \frac{1}{\psi(k-1)} .$$

Hence, by (6.2.7), for every $\varepsilon > 0$ we have, for a sequence of values of k tending to infinity

(6.2.11) $$\frac{\log \lambda_k}{\log \lambda_k - \log^+ \log^+ |a_k|} > \frac{\log \lambda_k}{\log (\log \frac{1}{\psi(k-1)})^{-1}} + o(1)$$

$$> (\theta - \varepsilon) + o(1).$$

Proceeding to limits and using (6.1.3), we get $1 + \rho_o \geq \theta$. This inequality is true for $\theta = \infty$ also, since, in that case, we get, in place of $(\theta - \varepsilon)$, an arbitrarily large number in (6.2.11), which gives $\rho_o = \infty$. This completes the proof of the theorem.

COROLLARY 1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U and have order $\rho_o (0 \leq \rho_o \leq \infty)$. Let $\{\lambda_{n_k}\}_{k=0}^{\infty}$ be the principal indices and $\rho(n_k)$ be the jump points of the central index of $f(z)$. Then,

(6.2.12) $$1 + \rho_o = \max \{1, U_o\}$$

where,

$$U_o = \limsup_{k \to \infty} \frac{\log \lambda_{n_k}}{-\log(1-\rho(n_k))} .$$

COROLLARY 2. Let $f(z) = 1 + \sum_{k=1}^{\infty} (r_1 r_2 \cdots r_k) z^{\lambda_k}$ $(r_k > 0, |z| < 1)$ be a nonconstant function such that for some $\beta (0 < \beta < \infty)$,

$$\log^+ r_k = O((\lambda_k - \lambda_{k-1}) \lambda_k^{-\beta}), \text{ as } k \to \infty$$

then $f(z) \in A$.

Essentially the hypotheses of Corollary 2 imply that $f(z)$ is analytic in the unit disc and by (6.2.6) it is of finite order. Hence, by corollary to Theorem 6.1, it belongs to the class A.

REMARK. The condition (6.2.8) is only sufficient and not necessary for the relation $1+\rho_0 = \max(1, \theta)$ to hold. This can be seen by the following:

(6.2.13) Example: Let,
$$f(z) = \sum_{k=0}^{\infty} k\, z^k + \sum_{k=0}^{\infty} \exp(k^{1/2})\, z^{2k}$$
$$= f_1(z) + f_2(z) \text{ (say)}$$

Then it is easily seen that $\log\ \mu(r,f_2) \simeq (8 \log \frac{1}{r})^{-1}$ as $r \to 1$. Since the coefficients in both the series are positive, we have

$$M(r,f_2) \le M(r,f_1+f_2) = M(r,f_1) + M(r,f_2) \le 2M(r,f_2).$$

Hence, $\log\ M(r,f) \sim \log\ M(r,f_2)$ as $r \to 1$ and therefore $f(z)$ and $f_2(z)$ have the same order ρ_0. Now, by (1.9.4),

$$\rho_0 = \limsup_{r \to \infty} \frac{\log\log \mu(r,f_2)}{-\log(1-r)} = 1.$$

Thus $f(z)$ is of order 1. It is also clear that $\psi(k)$ is not a nondecreasing function of k. Let,

$$A(k) = \frac{\log k}{-\log\log |a_k/a_{k-1}|}.$$

Then, $\lim_{k \to \infty} A(2k+1) = 0$ and $\lim_{k \to \infty} A(2k) = 2$. It is easily seen that $\theta = \limsup_{k \to \infty} A(k) = 2$. Hence $1 + \rho_0 = \max(1, \theta)$ is satisfied for the function $f(z)$.

6.3. We note that a coefficient formula analogous to (6.1.3) for the lower order ζ_0 does not always hold. For, consider the function given in (6.2.13). Then since $\log \mu(r, f_2) \approx (8 \log \frac{1}{r})^{-1}$, it follows by using (1.9.4) that $\rho_0 = \zeta_0 = 1$. But it can be easily seen that for this function

$$\liminf_{k \to \infty} \frac{\log^+ \log^+ |a_k|}{\log k} = 0.$$

In this section we obtain complete coefficient characterizations of the lower order ζ_0 which hold for every function of the class U. A result of L.R. Sons [75, Theorem 1] follows as a corollary to Theorem 6.5 proved below.

<u>THEOREM</u> 6.3. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ <u>belong to the class</u> U <u>and have the lower order</u> ζ_0 $(0 \leq \zeta_0 \leq \infty)$ <u>and let</u> $\{n_k\}_{k=1}^{\infty}$ <u>be an increasing sequence of natural numbers, then</u>

$$(6.3.1) \qquad 1 + \zeta_0 \geq \liminf_{k \to \infty} \frac{\log \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+ \log^+ |a_{n_k}|}.$$

<u>PROOF</u>. Let,

$$\limsup_{k \to \infty} \frac{\log \lambda_{n_k} - \log^+ \log^+ |a_{n_k}|}{\log \lambda_{n_{k-1}}} = \alpha.$$

Then $0 \leq \alpha \leq \infty$. We need only consider the case $0 \leq \alpha < 1$. For any ε such that $0 < \varepsilon \leq 1-\alpha$, we can find a fixed integer $N = N(\varepsilon)$ such that for $k > N$,

$$\log^+ |a_{r_k}| > \lambda_{n_k} \lambda_{n_{k-1}}^{-(\alpha+\varepsilon)} \quad . \tag{6.3.2}$$

Choose,

$$-\log r_k = \frac{1}{e} \lambda_{n_{k-1}}^{-(\alpha+\varepsilon)} , \quad k = 2,3,\ldots .$$

If $k > N$ and $r_k \leq r \leq r_{k+1}$, then by (6.3.2), using Cauchy's inequality

$$\begin{aligned} \log M(r) &\geq \log |a_{n_k}| + \lambda_{n_k} \log r \\ &\geq \log |a_{n_k}| + \lambda_{n_k} \log r_k \\ &> (1 - \frac{1}{e}) \lambda_{n_k} \lambda_{n_{k-1}}^{-(\alpha+\varepsilon)} . \end{aligned}$$

Therefore,

$$\begin{aligned} \log\log M(r) &> \log(1-e^{-1}) + \log \lambda_{n_k} - (\alpha+\varepsilon) \log \lambda_{n_{k-1}} \\ &= \log (1-e^{-1}) + (1 - \frac{1}{\alpha+\varepsilon}) - \frac{\log\log (1/r_{k+1})}{\alpha+\varepsilon} + \log\log (1/r_k) \\ &\geq \log (1-e^{-1}) + (1 - \frac{1}{\alpha+\varepsilon}) + (1 - \frac{1}{\alpha+\varepsilon}) \log\log (1/r) + o(1), \end{aligned}$$

and so,

$$\frac{\log\log M(r)}{-\log(1-r)} > o(1) + (1 - \frac{1}{\alpha+\varepsilon}) \frac{\log\log (1/r)}{-\log(1-r)} ,$$

which gives on proceeding to limits, $\zeta_o \geq \frac{1}{\alpha} - 1$, if $\alpha \neq 0$ and $\zeta_o = \infty$ if $\alpha = 0$. This establishes (6.3.1).

COROLLARY. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *belong to the class* U, *have order* $\rho_o (0 \leq \rho_o < \infty)$ *and lower order* ζ_o $(0 \leq \zeta_o < \infty)$. *If*

$$\text{(i)} \lim_{k \to \infty} \frac{\log \lambda_{k-1}}{\log \lambda_k} = 1 \quad \text{and} \quad \text{(ii)} \; S_o \equiv \lim_{k \to \infty} \frac{\log \lambda_k}{\log \lambda_k - \log^+ \log^+ |a_k|} \quad \text{exists,}$$

then $f(z)$ *is of regular growth, i.e.,* $\rho_o = \zeta_o = S_o$.

The corollary follows immediately in view of (6.1.3) and (6.3.1).

Our next theorem shows that, with $n_k = k$ and under an additional hypothesis, equality holds in (6.3.1).

THEOREM 6.4. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *belong to the class* U, *have order* ρ_o $(0 < \rho_o \leq \infty)$ *and lower order* ζ_o $(0 \leq \zeta_o \leq \infty)$ *and suppose* (6.2.8) *is satisfied, then*

$$(6.3.3) \qquad 1 + \zeta_o = \liminf_{k \to \infty} \frac{\log \lambda_{k-1}}{\log \lambda_k - \log^+ \log^+ |a_k|} .$$

PROOF. Let,

$$\limsup_{k \to \infty} \frac{\log \lambda_k - \log^+ \log^+ |a_k|}{\log \lambda_{k-1}} = \beta .$$

In view of Theorem 6.3, it is sufficient to prove that $\zeta_o \leq \frac{1}{\beta} - 1$.
Let $\psi(k)$ tend to C. It is easily seen that $\sum_{k=0}^{\infty} a_k z^{\lambda_k}$ has radius of convergence equal to C. Since $\rho_o > 0$, it follows that $C = 1$.
If $\psi(k) = C = 1$ for $k > k_o$, then it would follow that $\rho_o = 0$; thus $\psi(k) > \psi(k-1)$ for infinitely many values of k.

When $\psi(k) > \psi(k-1)$, the term $a_k z^{\lambda_k}$ becomes maximum term and we have

$$\mu(r) = |a_k| \, r^{\lambda_k}, \quad \nu(r) = \lambda_k \text{ for } \psi(k-1) \leq r < \psi(k).$$

Now, by (1.9.5), we have

$$1 + \zeta_o = \liminf_{r \to 1} \frac{\log \nu(r)}{-\log(1-r)} .$$

Suppose first that $\zeta_o < \infty$. Then for every ε such that $0 < \varepsilon < 1 + \zeta_o$, and for all r such that $R = R(\varepsilon) < r < 1$, $\nu(r) > (1-r)^{-(1+\zeta_o-\varepsilon)}$. Let $a_{k_1} z^{\lambda_{k_1}}$ and $a_{k_2} z^{\lambda_{k_2}}$ ($k_1 > k_o$ and $\psi(k_1-1) > R$) be two consecutive maximum terms so that $k_1 \leq k_2 - 1$. Suppose $k_1 < k \leq k_2$. Since $a_{k_1} z^{\lambda_{k_1}}$ is maximum term, we have $\nu(r) = \lambda_{k_1}$ for $\psi(k_1-1) \leq r < \psi(k_1)$. Hence for every r in this interval, $\lambda_{k_1} > (1-r)^{-(1+\zeta_o-\varepsilon)}$. This gives in particular,

$$\lambda_{k_1} > \left(\frac{1}{1-\psi(k_1)}\right)^{1+\zeta_o-\varepsilon} .$$

Further, we have

$$\psi(k_1) = \psi(k_1+1) = \ldots = \psi(k-1),$$

hence,

(6.3.4) $$\lambda_{k-1} \geq \lambda_{k_1} \geq \left(\frac{1}{1-\psi(k-1)}\right)^{1+\zeta_o-\varepsilon} .$$

Since,

$$\log |a_k| = \log |a_{k_o}| + \sum_{m=k_o+1}^{k} (\lambda_m - \lambda_{m-1}) \log (1/\psi(m-1))$$

$$\geq \log |a_{k_o}| + (\lambda_k - \lambda_{k_o}) \log (1/\psi(k-1)),$$

we get,

$$\log^+ \log^+ |a_k| \geq O(1) + \log (\lambda_k - \lambda_{k_o}) + \log \log (1/\psi(k-1)).$$

Or,

$$\frac{\log \lambda_k - \log^+ \log^+ |a_k|}{\log \lambda_{k-1}} \leq - \frac{\log \log (1/\psi(k-1))}{\log \lambda_{k-1}} + o(1)$$

$$\sim \frac{\log (1-\psi(k-1))^{-1}}{\log \lambda_{k-1}}, \text{ as } k \to \infty.$$

Using (6.3.4), this gives

$$\frac{\log \lambda_k - \log^+ \log^+ |a_k|}{\log \lambda_{k-1}} \leq \frac{1}{1+\zeta_o-\varepsilon} + o(1).$$

Thus, on proceeding to limits, we get $\beta \leq 1/(1+\zeta_o)$. This proves the theorem in case $\zeta_o < \infty$. When $\zeta_o = \infty$, above arguments with an arbitrarily large number in place of $(1 + \zeta_o - \varepsilon)$ give that $\beta = 0$. Hence the proof of the theorem is complete.

REMARKS. (i) (6.3.3) may fail to hold for functions of zero order. For, consider

$$(6.3.4) \qquad f(z) = \sum_{k=0}^{\infty} z^{(k!)^{k!}} .$$

The order ρ_o and lower order ζ_o of $f(z)$ are zero but

$$\liminf_{k \to \infty} \frac{(k-1)! \log (k-1)!}{k! \log k!} = 0.$$

(ii) In general the growth of the function $f(z)$ does not give any information about the analytic extension of the function outside the unit disc and vice-versa. For example, consider

$$f_1(z) = \sum_{k=0}^{\infty} \exp(k)\, z^{2^k} \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} z^k .$$

Then $f_1(z)$ and $f_2(z)$ are of the same order. In fact for both the functions $\rho_o = \zeta_o = 0$. But $f_1(z)$ cannot be continued analytically outside the unit disc, while $f_2(z)$ admits an analytic continuation.

(iii) The hypothesis (6.2.8) does not imply that $f(z)$ is of regular growth as can be seen by considering the following example of a function of irregular growth belonging to class $\mathcal{U}$.

(6.3.5) Example. Let,

$$n_1 = 4,\ n_{k+1} = n_k^4 , \quad (k = 1,2,\ldots.)$$

$$r_1 = r_2 = r_3 = e \quad , \quad r_m = \exp(m^{-1/2}) \quad \text{if} \quad n_k \le m < n_k^2$$

$$= \exp(n_{k+1}^{-1/2}) + \frac{\exp(m^{-1}) - \exp(n_{k+1}^{-1})}{n_{k+1}}$$

$$\text{if} \quad n_k^2 \le m < n_{k+1}$$

and let

$$f(z) = 1 + \sum_{k=1}^{\infty} (r_1 r_2 \ldots . r_k) z^k .$$

Then $\psi(k) = |a_{k-1}/a_k| = 1/r_k$ is an increasing function of k.

Let,

$$\theta(k) = \frac{\log k}{-\log\log r_k} .$$

Then, $\lim_{k \to \infty} \theta(n_k^2) = 1$ and $\lim_{k \to \infty} \theta(n_{k+1}) = 2$. It can be easily seen that $\limsup_{k \to \infty} \theta(k) = 2$ and $\liminf_{k \to \infty} \theta(k) = 1$, so that by (1.9.5), $\rho_o = 1$ and $\zeta_o = 0$. Hence $f(z)$ is of irregular growth.

(iv) (6.3.3) was earlier obtained [39] for $\lambda_k = k$.

We now remove the condition (6.2.8) from Theorem 6.4 and obtain a formula for the lower order which holds for every function of the class U. From this (1.9.6) can be derived as a particular case.

THEOREM 6.5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *belong to the class* U, *have order* $\rho_o (0 < \rho_o \leq \infty)$ *and lower order* ζ_o $(0 \leq \zeta_o \leq \infty)$, *then*

$$(6.3.6) \qquad 1 + \zeta_o = \max_{\{n_k\}} \left[\liminf_{k \to \infty} \frac{\log \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+ \log^+ |a_{n_k}|}\right],$$

where maximum in (6.3.6) *is taken over all increasing sequences* $\{n_k\}_{k=1}^{\infty}$ *of natural numbers.*

PROOF. Let $g(z) = \sum_{k=0}^{\infty} a_{n_k} z^{\lambda_{n_k}}$, $|z| < 1$, where $\{\lambda_{n_k}\}_{k=0}^{\infty}$ are the principal indices of $f(z)$. It is easily seen that $g(z)$ is analytic in the unit disc and that for every z in the unit disc, $f(z)$ and $g(z)$ have the same maximum term. Hence, by (1.9.4), the order and lower order of $g(z)$ are the same as those of $f(z)$. Thus $g(z)$ is of lower order ζ_o.

Further, since $\psi(n_k) = |a_{n_k}/a_{n_{k+1}}|^{1/(\lambda_{n_{k+1}} - \lambda_{n_k})}$ are jump points of the central index of $f(z)$, $\psi(n_k)$ is an increasing function of k. Therefore $g(z)$ satisfies the hypotheses of Theorem 6.4, and so by (6.3.3), we have

$$(6.3.7)\qquad 1+\zeta_o = \liminf_{k\to\infty} \frac{\log \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+\log^+|a_{n_k}|} .$$

But, from Theorem 6.3, we have

$$(6.3.8)\qquad 1+\zeta_o \geq \max_{\{n_h\}} \liminf_{h\to\infty} \frac{\log \lambda_{n_{h-1}}}{\log \lambda_{n_h} - \log^+\log^+|a_{n_h}|} .$$

Combining(6.3.7) and (6.3.8) we get (6.3.6). Hence the theorem.

COROLLARY. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U and have order ρ_o $(0 < \rho_o \leq \infty)$ and lower order ζ_o $(0 \leq \zeta_o \leq \infty)$, then

$$(6.3.9)\qquad 1+\zeta_o \leq (1+\rho_o) \liminf_{k\to\infty} \frac{\log \lambda_{k-1}}{\log \lambda_k} .$$

By (6.3.6), we have

$$1+\zeta_o \leq \left(\limsup_{k\to\infty} \frac{\log \lambda_k}{\log \lambda_k - \log^+\log^+|a_k|}\right)\left(\max_{\{n_k\}} \left[\liminf_{k\to\infty} \frac{\log \lambda_{n_{k-1}}}{\log \lambda_{n_k}}\right]\right) .$$

Now using (6.1.3), we get (6.3.9)

REMARKS. (i) (6.3.6) can be applied to obtain the lower order of the function given by (6.2.13). If we put

$$B(n_k) = \frac{\log n_{k-1}}{\log n_k - \log^+\log^+|a_{n_k}|}$$

then it can be easily seen that for $n_k = 2k$, $\lim_{k\to\infty} B(n_k) = 2$ and for

$n_k = 2k+1$, $\lim_{k \to \infty} B(n_k) = 1$. If $\{n_k\}$ is any other sequence of natural numbers, then $\liminf_{k \to \infty} B(n_k) \le 2$. Thus, $\max_{\{n_k\}} [\liminf_{k \to \infty} B(n_k)] = 2$, and so by (6.3.6), $\zeta_0 = 1$.

(ii) (6.3.9) was obtained by L.R. Sons [75] by a different method. In fact it implies that for functions of regular growth

$$\lim_{k \to \infty} \frac{\log \lambda_{k-1}}{\log \lambda_k} = 1.$$

The converse need not be true in general as can be seen by the example given in (6.3.5) in which $\lim_{k \to \infty} \frac{\log \lambda_{k-1}}{\log \lambda_k} = 1$ but the function is of irregular growth.

(iii) The function given by (6.3.4) shows that (6.3.6) may fail to hold for those functions of the class U which are of zero order.

<u>THEOREM</u> 6.6. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ <u>belong to the class</u> U <u>and have lower order</u> ζ_0 $(0 \le \zeta_0 \le \infty)$. <u>Let</u> $\{n_k\}_{k=1}^{\infty}$ <u>be an increasing sequence of natural numbers, then</u>

$$(6.3.10)\quad 1+\zeta_0 \ge \max \left\{1, \liminf_{k \to \infty} \frac{\log \lambda_{n_{k-1}}}{\log \left\{\dfrac{\lambda_{n_k} - \lambda_{n_{k-1}}}{\log^+ |a_{n_k}/a_{n_{k-1}}|}\right\}}\right\},$$

<u>quotient occuring at the right hand side of the above inequality being interpreted as zero for the values of</u> k <u>for which</u> $|a_{n_k}/a_{n_{k-1}}| \le 1$.

PROOF. Let,

$$\limsup_{k\to\infty} \frac{\log\left\{\dfrac{\lambda_{n_k}-\lambda_{n_{k-1}}}{\log^+|a_{n_k}/a_{n_{k-1}}|}\right\}}{\log \lambda_{n_{k-1}}} = \alpha .$$

It is sufficient to consider that $0 \leq \alpha < 1$. For any ε such that $0 < \varepsilon \leq 1 - \alpha$, we can choose a fixed integer $N = N(\varepsilon)$, such that for all $m > N$,

$$\text{(6.3.11)} \qquad \log^+ |a_{n_m}/a_{n_{m-1}}| > (\lambda_{n_m} - \lambda_{n_{m-1}})\, \lambda_{n_{m-1}}^{-(\alpha+\varepsilon)} .$$

Since,

$$|a_{n_k}| = |a_{n_N}| \prod_{m=N+1}^{k} |a_{n_m}/a_{n_{m-1}}|,$$

using (6.3.11) with the fact that the right hand side of this inequality is positive, we have

$$\text{(6.3.12)} \quad \log |a_{n_k}| > \log |a_{n_N}| + \sum_{m=N+1}^{k} (\lambda_{n_m}-\lambda_{n_{m-1}})\, \lambda_{n_{m-1}}^{-(\alpha+\varepsilon)}$$

$$> \log |a_{n_N}| + (\lambda_{n_k} - \lambda_{n_N})\, \lambda_{n_{k-1}}^{-(\alpha+\varepsilon)} .$$

Let, $-\log r_k = \frac{1}{e} \; \lambda_{n_{k-1}}^{-(\alpha+\varepsilon)}$, $k = 2,3,\ldots.$ If $k > N$ and $r_k \leq r \leq r_{k+1}$, then by (6.3.12), using Cauchy's estimate,

$$\log M(r) \geq \log |a_{n_k}| + \lambda_{n_k} \log r_k$$

$$> \log |a_{n_N}| + (1-e^{-1})\lambda_{n_k} \lambda_{n_{k-1}}^{-(\alpha+\varepsilon)} - \lambda_{n_N} \lambda_{n_{k-1}}^{-(\alpha+\varepsilon)} .$$

Therefore,

$$\log\log M(r) > \log \lambda_{n_k} - (\alpha+\varepsilon) \log \lambda_{n_{k-1}} + O(1)$$

$$= - \frac{\log\log (1/r_{k+1})}{\alpha + \varepsilon} + \log\log (1/r_k) + O(1)$$

$$\geq (1 - \frac{1}{\alpha+\varepsilon}) \log\log (1/r) + O(1),$$

and so,

$$\frac{\log\log M(r)}{-\log(1-r)} > o(1) + (1 - \frac{1}{\alpha+\varepsilon}) \frac{\log\log (1/r)}{-\log(1-r)} ,$$

which on proceeding to limits gives

$$\zeta_o \geq \frac{1}{\alpha} - 1 \text{ if } \alpha \neq 0 \text{ and } \zeta_o = \infty \text{ if } \alpha = 0.$$

This establishes (6.3.10).

COROLLARY. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U, have order $\rho_o (0 \leq \rho_o < \infty)$ and lower order $\zeta_o (0 \leq \zeta_o < \infty)$. If (i) $\lim_{k\to\infty} \frac{\log \lambda_{k-1}}{\log \lambda_k} = 1$, (ii) $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)} \geq 1/e$ and

(iii) $S \equiv \lim_{k\to\infty} \dfrac{\log \lambda_k}{\log \left(\dfrac{\lambda_k - \lambda_{k-1}}{\log^+ |a_k/a_{k-1}|}\right)}$ exists, then $f(z)$ is of regular growth and $\rho_o = \zeta_o = S$.

The Corollary follows immediately by using Theorem 6.6 and (6.2.6).

REMARK. An alternative proof of the above theorem can also be given by using Theorem 6.3 as follows.

From (6.3.12), we obtain for all $k > N$,

$$\log^+ |a_{n_k}| > \log |a_{n_N}| + (\lambda_{n_k} - \lambda_{n_N})\, \lambda_{n_{k-1}}^{-(\alpha+\varepsilon)} .$$

Or,

$$\log^+ \log^+ |a_{n_k}| > \log \lambda_{n_k} - (\alpha+\varepsilon) \log \lambda_{n_{k-1}} + O(1),$$

which implies,

$$\frac{\log \lambda_{n_k} - \log^+\log^+ |a_{n_k}|}{\log \lambda_{n_{k-1}}} < \alpha + \varepsilon - o(1).$$

Now, on proceeding to limits and using Theorem 6.3, this gives $1/(1 + \zeta_o) \leq \alpha+\varepsilon$. Therefore, $1 + \zeta_o \geq 1/\alpha$ if $\alpha \neq 0$ and $\zeta_o = \infty$ if $\alpha = 0$. This proves (6.3.10).

THEOREM 6.7. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *belong to the class* U , *have order* ρ_o $(0 < \rho_o \leq \infty)$, *lower order* ζ_o $(0 \leq \zeta_o \leq \infty)$, *and satisfy* (6.2.8), *then*

$$(6.3.13) \qquad 1 + \zeta_o = \max\left(1, \liminf_{k \to \infty} \frac{\log \lambda_{k-1}}{\log \left\{\dfrac{\lambda_k - \lambda_{k-1}}{\log|a_k/a_{k-1}|}\right\}}\right) .$$

PROOF. First let $\zeta_o < \infty$. Then for every ε such that $0 < \varepsilon < 1 + \zeta_o$ and for all $k > k_o = k_o(\varepsilon)$, we have, as in the proof of Theorem 6.4, that

$$\lambda_{k-1} \geq (1 - \psi(k-1))^{-(1+\zeta_o-\varepsilon)} .$$

Therefore, on proceeding to limits after some simple calculations, we get

$$1 + \zeta_o \leq \liminf_{k\to\infty} \frac{\log \lambda_{k-1}}{-\log (1-\psi(k-1))}$$

$$= \liminf_{k\to\infty} \frac{\log \lambda_{k-1}}{\log \left(\frac{\lambda_k - \lambda_{k-1}}{\log|a_k/a_{k-1}|}\right)} .$$

If $\zeta_o = \infty$, then proceeding as above with an arbitrarily large number in place of $(1 + \zeta_o - \varepsilon)$, we get

$$\liminf_{k\to\infty} \frac{\log \lambda_{k-1}}{\log \left(\frac{\lambda_k - \lambda_{k-1}}{\log|a_k/a_{k-1}|}\right)} = \infty .$$

Hence, in view of Theorem 6.6 the proof of (6.3.13) is complete.

COROLLARY. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *belong to the class* U, *have order* ρ_o $(0 < \rho_o \leq \infty)$ *and lower order* ζ_o $(0 \leq \zeta_o \leq \infty)$. *Let* $\{\lambda_{n_k}\}_{k=0}^{\infty}$ *be the principal indices and* $\rho(n_k)$ *be the jump points of the central index of* $f(z)$. *Then*,

$$(6.3.14) \qquad 1 + \zeta_o = \max \left(1, \liminf_{k\to\infty} \frac{\log \lambda_{n_{k-1}}}{-\log (1-\rho(n_k))}\right) .$$

We now obtain another formula involving the coefficients of the power series of $f(z)$. This formula also holds for every function of the class $\mathcal{U}$, having positive order.

THEOREM 6.8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ belong to the class $\mathcal{U}$ and have order ρ_0 $(0 < \rho_0 \leq \infty)$ and lower order ζ_0 $(0 \leq \zeta_0 \leq \infty)$, then

$$(6.3.15) \qquad 1 + \zeta_0 = \max_{\{n_k\}} \left[\liminf_{k \to \infty} \frac{\log \lambda_{n_{k-1}}}{\log \left\{ \dfrac{\lambda_{n_k} - \lambda_{n_{k-1}}}{\log^+ |a_{n_k}/a_{n_{k-1}}|} \right\}}\right],$$

the quotient occuring at the right hand side being interpreted to be zero for those values of k for which $|a_{n_k}/a_{n_{k-1}}| \leq 1$.

PROOF. Let $\{\lambda_{n_k}\}_{k=0}^{\infty}$ be principal indices and $\rho(n_k)$ be the jump points of the central index of $f(z)$. It is clear that $\rho(n_k) \to 1$ as $k \to \infty$. Further,

$$0 \leq \rho(n_k) < \rho(n_{k+1}) < 1, \quad k = 1,2,\ldots$$

and

$$\nu(r) = \lambda_{n_k} \text{ for } \rho(n_k) \leq r < \rho(n_{k+1}).$$

Also since $a_{n_k} z^{\lambda_{n_k}}$ and $a_{n_{k+1}} z^{\lambda_{n_{k+1}}}$ are consecutive maximum terms, we have

$$\rho(n_{k+1}) = |a_{n_k}/a_{n_{k+1}}|^{1/(\lambda_{n_{k+1}} - \lambda_{n_k})}.$$

Hence, by (1.9.5),

(6.3.16) $$1 + \zeta_o = \liminf_{k \to \infty} \frac{\log \lambda_{n_{k-1}}}{\log \{\frac{\lambda_{n_k} - \lambda_{n_{k-1}}}{\log^+ |a_{n_k}/a_{n_{k-1}}|}\}}.$$

But by Theorem 6.6,

(6.3.17) $$1 + \zeta_o \geq \max_{\{n_h\}} [\liminf_{h \to \infty} \frac{\log \lambda_{n_{h-1}}}{\log \{\frac{\lambda_{n_h} - \lambda_{n_{h-1}}}{\log^+ |a_{n_h}/a_{n_{h-1}}|}\}}].$$

Comparing (6.3.16) and (6.3.17), we get (6.3.15) and the proof of the theorem is complete.

6.4. Let $\mathcal{U}_R$ denote the class of functions analytic in the disc $D_R = \{z: |z| < R\}$, $0 < R \leq \infty$, which are not polynomials or rational functions. It is well known ([42] , [48] , [51] , [91]) that there exist functions in the class $\mathcal{U}_R$ whose rates of growth (as measured by their maximum modulii) are arbitrarily fast or arbitrarily slow. For $R = \infty$, Iepson [43] , proved the following by means of simple constructions:

THEOREM A. *Let $h(r)$ and $k(r)$ be positive functions of r for $r > 0$ such that $\log k(r) \neq O(\log r)$ as $r \to \infty$. Then there exists a function $f(z)$ with nonnegative coefficients, belonging to $\mathcal{U}_\infty$, and two sequences $\{c_n\}$ and $\{r_n\}$ of positive numbers tending monotonically to infinity such that for every positive integer n, $M(c_n,f) > h(c_n)$ and $M(r_n,f) < k(r_n)$.*

THEOREM B. Let $h(r)$ be a positive function of r for $r > 0$ bounded in every finite interval. Then there exists a function $f(z)$ with nonnegative coefficients, belonging to $\mathcal{U}_\infty$, such that $M(r) > h(r)$ for all r.

THEOREM C. Let $k(r)$ be a positive function of r having a positive lower bound for $r > 0$ and such that $(\log k(r)/\log r) \to \infty$ as $r \to \infty$. Then there exists a function $f(z)$ with nonnegative coefficients, belonging to $\mathcal{U}_\infty$, such that $M(r) < k(r)$ for all r.

In the present section, using the above results, we prove the existence of similar functions in the class $\mathcal{U}_R$ $(0 < R < \infty)$. Theorem 6.9 shows the existence of a function belonging to $\mathcal{U}_R$ $(0 < R < \infty)$ whose upper rate of growth is arbitrarily fast and simultaneously whose lower rate of growth is arbitrarily slow. In Theorem 6.12 we show that there exists a function in the class $\mathcal{U}_R$ $(0 < R < \infty)$ which has two arbitrarily prescribed rates of growth on two different, although unspecified, sequences tending to R. As corollaries to Theorems 6.9 to 6.12 we also show the existence of similar functions belonging to the class $\mathcal{U}_R^C$ consisting of those functions which are analytic in $D_R^C \equiv \{z : |z| > R\}$, $(0 < R < \infty)$.

THEOREM 6.9. Let $\lambda(r)$ and $\mu(r)$ be positive functions of r for $0 < r < R < \infty$ such that $\log \lambda(r) \neq O(-\log(R-r))$ as $r \to R^-$. Then there exists a function $F(z)$ belonging to class $\mathcal{U}_R$ $(0 < R < \infty)$, with nonnegative coefficients, and two sequences $\{s_n\}$ and $\{t_n\}$ of positive numbers tending monotonically to R such that for every positive integer n, $M(s_n, F) > \mu(s_n)$ and $M(t_n, F) < \lambda(t_n)$.

PROOF. Define,

$$k(s) = \lambda(R - \frac{1}{s}) \quad \text{for } 0 < s < \frac{1}{R}$$
$$= 0 \quad \text{for } \frac{1}{R} \le s < \infty$$

and

$$h(s) = \mu(R - \frac{1}{s}) \quad \text{for } 0 < s < \frac{1}{R}$$
$$= 0 \quad \text{for } \frac{1}{R} \le s < \infty .$$

Then $h(s)$ and $k(s)$ are positive functions of s for $0 < s < \infty$ and $\log k(s) = \log \lambda(R - \frac{1}{s}) \neq 0 (\log s)$ as $s \to \infty$. Therefore by Theorem A, there exists a function $\phi(z)$ of the class $\mathcal{U}_\infty$ with nonnegative coefficients and two sequences $\{c_n\}$ and $\{r_n\}$ of positive numbers, tending to infinity, such that for every $n > 0$,

$$\phi(r_n) = M(r_n, \phi) < k(r_n),$$

and

$$\phi(c_n) = M(c_n, \phi) > h(c_n) .$$

Let N be a positive integer such that r_N and c_N are greater than or equal to $1/R$, and set

$$t_n = R - \frac{1}{r_{n+N}}, \; s_n = R - \frac{1}{c_{n+N}}, \; n = 1,2,\ldots\ldots .$$

Then $\{t_n\}$ and $\{s_n\}$ are sequences of positive numbers tending monotonically to R. Define,

$$F(z) = \phi(\frac{1}{R-z}) \quad \text{for} \quad z \in D_R .$$

Then $F(z) \in \mathcal{U}_R$ and has nonnegative coefficients. Further,

$$F(t_n) = M(t_n,F) = M(\frac{1}{R-t_n},\phi) = M(r_{n+N},\phi) < k(r_{n+N}) = k(\frac{1}{R-t_n}) = \lambda(t_n)$$

and

$$F(s_n) = M(s_n,F) = M(\frac{1}{R-s_n},\phi) = M(c_{n+N},\phi) > h(c_{n+N}) = h(\frac{1}{R-s_n}) = \mu(s_n).$$

Hence the theorem.

COROLLARY. *Let $\lambda_1(r)$ and $\mu_1(r)$ be positive functions of r for $0 < R < r < \infty$ such that $\log \lambda_1(r) \neq 0\,(-\log(\frac{1}{R}-\frac{1}{r}))$ as $r \to R^+$. Then there exists a function $G(z)$ belonging to class $\mathcal{U}_R^C$ $(0 < R < \infty)$, with nonnegative coefficients and two sequences $\{s_n'\}$ and $\{t_n'\}$ of positive numbers decreasing monotonically to R such that for every positive integer n, $M(s_n', G) > \mu_1(s_n')$ and $M(t_n', G) < \lambda_1(t_n')$.*

Let $S = \frac{1}{R}$ and $u = \frac{1}{r}$. Define $\lambda(u) = \lambda_1(1/u)$ and $\mu(u) = \mu_1(1/u)$. Then $\log \lambda(u) \neq 0\,(-\log(S-u))$. Hence there exist two sequences $\{s_n\}$ and $\{t_n\}$ tending to S, and a function $F(z) \in \mathcal{U}_S$ such that

$$M(t_n,F) < \lambda(t_n) \quad \text{and} \quad M(s_n,F) > \mu(s_n).$$

Put $s_n' = 1/s_n$, $t_n' = 1/t_n$ and $G(z) = F(1/z)$. Then

$$M(s_n',G) = M(\frac{1}{s_n'}, F) = M(s_n,F) > \mu(s_n) = \mu(1/s_n') = \mu_1(s_n')$$

and

$$M(t_n',G) = M(\frac{1}{t_n'},F) = M(t_n,F) < \lambda(t_n) = \lambda(1/t_n') = \lambda_1(t_n').$$

THEOREM 6.10. Let $\mu(r)$ be a positive function of r for $0 < r < R < \infty$ such that it is bounded in every subinterval of $(0,R)$. Then, there exists a function $F(z)$, belonging to the class $\mathcal{U}_R$ $(0 < R < \infty)$ with nonnegative coefficients, such that $M(r,F) > \mu(r)$ for $0 < r < R < \infty$.

PROOF. Let $h(s)$ be as defined in Theorem 6.9. Then, $h(s)$ is a positive function of s for $0 < s < \infty$ and is bounded in every finite interval. Hence, by Theorem B, there exists a function $\phi(z)$ of class $\mathcal{U}_\infty$ having nonnegative coefficients such that $M(s,\phi) > h(s)$ for all s. Set, $s = \frac{1}{R-r}$ and $F(z) = \phi\left(\frac{1}{R-z}\right)$, $|z| < R$. Then $F(z) \in \mathcal{U}_R$ and has nonnegative coefficients. Further for every r such that $0 < r < R < \infty$,

$$F(r) = M(r,F) = M\left(\frac{1}{R-r}, \phi\right) = M(s,\phi) > h(s) = h\left(\frac{1}{R-r}\right) = \mu(r).$$

Hence the theorem.

COROLLARY. Let $\mu_1(r)$ be a positive function of r for $0 < R < r < \infty$ such that it is bounded on every finite subinterval of (R, ∞). Then there exists a function $G(z)$ belonging to the class $\mathcal{U}_R^C$ $(0 < R < \infty)$, with nonnegative coefficients such that $M(r,F) > \mu_1(r)$ for $0 < R < r < \infty$.

THEOREM 6.11. Let $\lambda(r)$ be a positive function of r for $0 < r < R < \infty$ such that

$$\lim_{r \to R^-} \frac{\log \lambda(r)}{-\log (R-r)} = \infty .$$

Then, there exists a function $F(z)$ belonging to $\mathcal{U}_R$, $(0 < R < \infty)$, with nonnegative coefficients such that $M(r,F) < \lambda(r)$ for $0 < r < R < \infty$.

PROOF. Let $k(s)$ be defined as in Theorem 6.9. Then the condition on $\lambda(r)$ implies

$$\lim_{s \to \infty} \frac{\log k(s)}{\log s} = \infty .$$

Hence by theorem C, there exists a function $\phi(z) \varepsilon\ \mathcal{U}_\infty$ and having nonnegative coefficients such that for all s,

$$\phi(s) = M(s, \phi) < k(s).$$

Define $F(z) = \phi\left(\frac{1}{R-z}\right)$, $|z| < R$ and $s = \frac{1}{R-r}$. Then as above $F(z) \varepsilon\ \mathcal{U}_R$ and has nonnegative coefficients. Further for every r such that $0 < r < R < \infty$, we have

$$F(r) = M(r,F) = M\left(\frac{1}{R-r}, \phi\right) = M(s, \phi) < k(s) = k\left(\frac{1}{R-r}\right) = \lambda(r).$$

Hence the theorem.

COROLLARY. Let $\lambda_1(r)$ be a positive function of r for $0 < R < r < \infty$ such that

$$\lim_{r \to R^+} \frac{\log \lambda(r)}{\log\left(\frac{1}{R} - \frac{1}{r}\right)^{-1}} = \infty .$$

Then there exists a function $G(z) \varepsilon\ \mathcal{U}_R^C$ $(0 < R < \infty)$, having nonnegative coefficients such that $M(r,G) < \lambda_1(r)$ for $0 < R < r < \infty$,

THEOREM 6.12. Let $\lambda(r)$ and $\mu(r)$ be positive and continuous functions of r for $0 < r < R < \infty$ such that $\log \lambda(r) \neq 0\,(-\log(R-r))$ and $\log \mu(r) \neq O(-\log(R-r))$ as $r \to R$. Then there exists a function $F(z)$ belonging to the class $\mathcal{U}_R(0 < R < \infty)$, with nonnegative coefficients such that $M(r,f) = \lambda(r)$ on some sequence of values of $r \to R$ and $M(r,f) = \mu(r)$ on another such sequence.

PROOF. Let $\alpha(r) = \min(\mu(r), \lambda(r))$ and $\beta(r) = \max(\mu(r), \lambda(r))$. First, let $\log \alpha(r) \neq O(-\log(R-r))$ as $r \to R$. Then by Theorem 6.9 there exists a function $F(z)$ of the class $\mathcal{U}_R$ $(0 < R < \infty)$ with nonnegative coefficients and two increasing sequences $\{s_n\}$ and $\{t_n\}$ of positive numbers tending to R such that for every positive integer n, $M(s_n,F) > \beta(s_n)$ and $M(t_n,F) < \alpha(t_n)$. Hence $M(s_n,F) > \mu(s_n)$, $M(t_n,F) < \mu(t_n)$ and $M(s_n,F) > \lambda(s_n)$, $M(t_n,F) < \lambda(t_n)$. Further, by the continuity of $M(r)$, $\lambda(r)$ and $\mu(r)$ it follows that $M(r,F) = \mu(r)$ on one sequence tending to R and $M(r,F) = \lambda(r)$ on another such sequence.

Next suppose that $\log \alpha(r) = O(-\log(R-r))$ as $r \to R$. We can find a pair $\mu^*(r)$ and $\lambda^*(r)$ of continuous functions on $0 < r < R < \infty$, which are bounded away from zero,

$$\lim_{r \to R} \frac{\log \mu^*(r)}{-\log(R-r)} = \lim_{r \to R} \frac{\log \lambda^*(r)}{-\log(R-r)} = \infty ,$$

and there exists two sequences $\{s_n^*\}$ and $\{t_n^*\}$ such that

$$\mu(s_n^*) = \mu^*(s_n^*) \quad \text{and} \quad \lambda(t_n^*) = \lambda^*(t_n^*).$$

Let $\delta(r) = \min(\mu^*(r), \lambda^*(r))$. Then by Theorem 6.11, there exists a function $F(z) \in \mathcal{U}_R$ with nonnegative coefficients such that $M(r,F) < \delta(r)$ for $0 < r < R < \infty$. Now,

$$\lim_{n \to \infty} \frac{\log \mu(s_n^*)}{-\log(R-s_n^*)} = \lim_{n \to \infty} \frac{\log \lambda(t_n^*)}{-\log(R-t_n^*)} = \infty.$$

But,

$$\frac{\log \alpha(s_n^*)}{-\log(R-s_n^*)} \text{ and } \frac{\log \alpha(t_n^*)}{-\log(R-t_n^*)}$$

are bounded as $n \to \infty$, hence

$$\frac{\log \mu(t_n^*)}{-\log(R-t_n^*)} \quad \text{and} \quad \frac{\log \lambda(s_n^*)}{-\log(R-s_n^*)}$$

are also bounded. Since $F(z)$ is not a rational function

$$\lim_{n \to \infty} \frac{\log M(s_n^*,F)}{-\log(R-s_n^*)} = \lim_{n \to \infty} \frac{\log M(t_n^*,F)}{-\log (R-t_n^*)} = \infty .$$

Therefore, $M(s_n^*) > \lambda(s_n^*)$ and $M(t_n^*) > \mu(t_n^*)$. But

$$M(s_n^* , F) < \delta (s_n^*) \leq \mu^*(s_n^*) = \mu(s_n^*)$$

and

$$M(t_n^*, F) < \delta (t_n^*) \leq \lambda^* (t_n^*) = \lambda(t_n^*) .$$

Now using the continuity of the maximum modulus $M(r)$ and the functions $\lambda(r)$ and $\mu(r)$, we get $M(r,F) = \mu(r)$ on one sequence tending to R and $M(r,F) = \lambda(r)$ on another such sequence.

<u>COROLLARY</u>. <u>Let</u> $\lambda_1(r)$ <u>and</u> $\mu_1(r)$ <u>be positive and continuous functions of</u> r <u>for</u> $0 < R < r < \infty$ <u>such that</u> $\log \lambda_1(r) \neq 0(-\log (\frac{1}{R} - \frac{1}{r}))$ <u>and</u> $\log \mu_1(r) \neq 0 (-\log (\frac{1}{R} - \frac{1}{r}))$ <u>as</u> $r \to R^+$. <u>Then there exists a function</u> $G(z)$ <u>belonging to</u> $\mathcal{U}_R^C$ $(0 < R < \infty)$ <u>with nonnegative coefficients such that</u> $M(r,G) = \lambda_1(r)$ <u>on some sequence of values of</u> r <u>decreasing to</u> R <u>and</u> $M(r,G) = \mu_1(r)$ <u>on another such sequence</u>.

CHAPTER 7

COEFFICIENTS OF FUNCTIONS ANALYTIC IN THE UNIT DISC

7.1. In this chapter we continue to study the growth of functions of class U in relation to their coefficients. If two functions in U are of same nonzero finite order, a precise comparison of their growth is not possible by confining to the notion of order only. For this purpose, we introduce the notions of type and lower type as below.

DEFINITION 7.1. *A function* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *of the class* U *having order* ρ_o $(0 < \rho_o < \infty)$ *is said to be of type* T_o *and lower type* t_o *if*

$$(7.1.1) \qquad \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r)}{(1-r)^{-\rho_o}} = \begin{matrix} T_o \\ t_o \end{matrix} \qquad (0 \le t_o \le T_o \le \infty).$$

The function $f(z)$ is said to have growth $\{\rho_o , T_o\}$ if it is of order not exceeding ρ_o and type not exceeding T_o if of order ρ_o. A function $f(z) \in U$ and having regular growth is said to be of perfectly regular growth if $T_o = t_o < \infty$.

The study of the central index of a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ belonging to the class U is facilitated by introducing the concept of growth numbers. Thus, we have

DEFINITION 7.2. *A function* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *of the class* U *having order* ρ_o $(0 < \rho_o < \infty)$ *is said to be of growth number* μ_o *and lower growth number* δ_o *if*

$$(7.1.2)\qquad \lim_{r\to 1}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\nu(r)}{r(1-r)^{-\rho_o-1}} = \begin{matrix}\mu_o\\ \delta_o\end{matrix}\quad (0 \le \delta_o \le \mu_o \le \infty).$$

In this chapter we first obtain a complete coefficient characterization for the type of the functions of the class U. Theorem 7.4 gives a coefficient formula for the lower type of those functions in U which satisfy (6.2.8) and the relation $\lambda_{k-1} \sim \lambda_k$ as $k \to \infty$. Theorem 7.5 gives another coefficient formula for the lower type which holds for a wider subclass of U. In Section 7.4 we give coefficient equivalents for the growth numbers, which are valid for functions of the class U satisfying (6.2.8). We also prove a decomposition theorem involving type and lower type. In the last section we obtain some relations involving type and lower type of $f(z) \in U$ and the ratio of the consecutive coefficients in its Taylor series expansion.

We shall suppose throughout in this chapter that the Taylor series expansion for a function of the class U is given by (6.1.1).

7.2. We first prove a lemma which will be required in the sequel.

LEMMA 7.1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U and have order ρ_o $(0 < \rho_o < \infty)$, type $T_o (0 \le T_o \le \infty)$ and lower type $t_o (0 \le t_o \le \infty)$, then

$$(7.2.1)\qquad \begin{matrix}T_o\\ t_o\end{matrix} = \lim_{r\to 1}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log \mu(r)}{(1-r)^{-\rho_o}}$$

where

$$\mu(r) = \max_{k \ge 0} \{|a_k|\, r^{\lambda_k}\}.$$

PROOF. By (1.9.3), for all r such that $0 < r < 1$ we have

$$\log M(r) < \log \mu(r) + \log [\{2\nu(r + \frac{1-r}{\nu(r)}) + 1\} \frac{1}{1-r}],$$

and from (1.9.5) it follows that for given $\varepsilon > 0$ and for all r such that $1 > r > r_o = r_o(\varepsilon) > 0$,

$$\nu(r) < (1-r)^{-(1+\rho_o+\varepsilon)} .$$

Hence, for $0 < r_o < r < 1$,

$$\log M(r) < \log \mu(r) + (1+\rho_o+\varepsilon) \log \frac{\nu(r)}{\nu(r)-1} - (2+\rho_o+\varepsilon)\log (1-r) + O(1).$$

Dividing both sides of this inequality by $(1-r)^{-\rho_o}$ and proceeding to limits, we get,

$$\lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r)}{(1-r)^{-\rho_o}} \leq \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \mu(r)}{(1-r)^{-\rho_o}} .$$

Since $\mu(r) \leq M(r)$, the reverse inequalities follow . Hence the lemma.

For the function $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$, set

$$\text{(7.2.2)} \qquad v = \limsup_{k \to \infty} \frac{(\log^+ |a_k|)^{\rho_c+1}}{\lambda_k^{\rho_o}} , \quad (0 < \rho_o < \infty) .$$

Then, we have the following theorem.

THEOREM 7.1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *belong to the class* U. *If* $0 < v < \infty$, *the function* $f(z)$ *is of order* ρ_o *and type* T_o, *if and only if*, $v = \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T_o$. *If* $v = 0$ *or* ∞, $f(z)$ *is respectively of growth* $\{\rho_o, 0\}$ *or of growth not less than* $\{\rho_o, \infty\}$, *and conversely*.

PROOF. Let $v < \infty$. For given $\varepsilon > 0$ and for all $k > N = N(\varepsilon)$ we have by (7.2.2),

$$\log^+ |a_k| < (v+\varepsilon)^{1/(\rho_o+1)} \lambda_k^{\rho_o/(\rho_o+1)} .$$

Therefore,

$$\frac{\log^+ \log^+ |a_k|}{\log \lambda_k} < \frac{\log (v+\varepsilon)}{(\rho_o+1) \log \lambda_k} + \frac{\rho_o}{\rho_o+1} .$$

Hence, by (6.1.3), $f(z)$ is of order at most ρ_o. Similarly if $v > 0$, the order of $f(z)$ is at least ρ_o. Thus, if $0 < v < \infty$, $f(z)$ is of order ρ_o.

Let $0 \leq T_o < \infty$. For given $\varepsilon > 0$ and for all r such that $1 > r > r_o = r_o(\varepsilon) > 0$, (7.1.1) gives,

$$\log M(r) < (T_o + \varepsilon) (1 - r)^{-\rho_o} .$$

Using Cauchy's estimate, we have for all k and for all r such that $1 > r > r_o = r_o(\varepsilon) > 0$,

$$\log |a_k| < (T_o+\varepsilon) (1-r)^{-\rho_o} - \lambda_k \log r.$$

Since the right hand side of the above inequality is a positive quantity for $r < 1$, we have

(7.2.3) $$\log^+ |a_k| < (T_o+\varepsilon)\,(1-r)^{-\rho_o} - \lambda_k \log r.$$

Choose, $(1-r)^{-1} = (\lambda_k/(T_o+\varepsilon)\rho)^{1/(\rho_o+1)}$. Then, for all $k > k_o = k_o(r_o)$, (7.2.3) gives

$$\log^+ |a_k| < \frac{(T_o+\varepsilon)^{1/(\rho_o+1)}\,\lambda_k^{\rho_o/(\rho_o+1)}}{\rho_o^{\rho_o/(\rho_o+1)}}\,(1+\rho_o) + o(1).$$

Therefore,

$$\frac{(\log^+ |a_k|)^{\rho_o+1}}{\lambda_k^{\rho_o}} < \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\,(T_o+\varepsilon) + o(1).$$

Now, on proceeding to limits,

(7.2.4) $$v \leq \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\,T_o\,.$$

Next suppose $0 \leq v < \infty$. For given $\varepsilon > 0$ and for all $k > k_o = k_o(\varepsilon)$, by (7.2.2)

$$\log^+ |a_k| < \lambda_k^{\rho_o/(\rho_o+1)}\,(v+\varepsilon)^{1/(\rho_o+1)}.$$

Let us write,

$$\therefore \quad M(r) \le \sum_{k=0}^{\infty} |a_k| \, r^{\lambda_k}$$

$$(7.2.5) \qquad = \sum_{k=0}^{k_o} |a_k| \, r^{\lambda_k} + \sum_{k=k_o+1}^{N} |a_k| \, r^{\lambda_k} + \sum_{k=N+1}^{\infty} |a_k| \, r^{\lambda_k},$$

where

$$(7.2.6) \qquad N = \left[\frac{2^{(\rho_o+1)} (v+\varepsilon)}{(-\log r)^{\rho_o+1}} \right].$$

Now,

$$\sum_{k=N+1}^{\infty} \exp\{\lambda_k^{\rho_o/(\rho_o+1)} (v+\varepsilon)^{1/(\rho_o+1)}\} r^{\lambda_k} \le \sum_{k=N+1}^{\infty} r^{\lambda_k/2} \le \frac{r^{(N+1)/2}}{1-r^{1/2}}.$$

Since, as $r \to 1$,

$$(7.2.7) \qquad (1 - r) = - \log r \; [1 + O(- \log r)] \; ,$$

therefore,

$$\log (1-r^{1/2}) = - \log 2 + \log \log \frac{1}{r} + O(-\log r),$$

and so,

$$\log \frac{r^{(N+1)/2}}{1-r^{1/2}} = - \frac{2^{\rho_o}(v+\varepsilon)}{(-\log r)^{\rho_o}} + \log 2 - \log \log \frac{1}{r} + O(-\log r)$$

$$= -2^{\rho_o} (v+\varepsilon) \, (-\log r)^{-\rho_o} \, \{1 + o(1)\}.$$

Thus,

$$(7.2.8) \qquad \sum_{k=N+1}^{\infty} \exp\{\lambda_k^{\rho_o/(\rho_o+1)} (v+\varepsilon)^{1/(\rho_o+1)}\} r^{\lambda_k} \leq \frac{r^{(N+1)/2}}{1-r^{1/2}} = o(1) .$$

Let,

$$g(\lambda_k, r) = \lambda_k^{\rho_o/(\rho_o+1)} (v+\varepsilon)^{1/(\rho_o+1)} + \lambda_k \log r.$$

Then, it can be easily seen that for all k and for all r $(0 < r < 1)$,

$$g(\lambda_k, r) \leq \frac{\rho_o^{\rho_o} (v+\varepsilon)}{(\rho_o+1)^{\rho_o+1} (-\log r)^{\rho_o}} .$$

Hence, by (7.2.5) and (7.2.8), we get

$$M(r) < A(k_o) + N \exp\{\frac{\rho_o^{\rho_o} (v+\varepsilon)}{(\rho_o+1)^{\rho_o+1} (-\log r)^{\rho_o}}\} + o(1),$$

where $A(k_o)$ is a constant. Using (7.2.6) and (7.2.7), we get from the last inequality, that

$$\frac{\log M(r)}{(1-r)^{-\rho_o}} < o(1) + \frac{\rho_o^{\rho_o} (v+\varepsilon)}{(\rho_o+1)^{\rho_o+1}} .$$

Now, on proceeding to limits, this gives

$$(7.2.9) \qquad T_o \leq v \frac{\rho_o^{\rho_o}}{(\rho_o+1)^{\rho_o+1}} .$$

On combining (7.2.4) and (7.2.9) and using the fact that for $0 < v < \infty$, $f(z)$ is of order ρ_o, we get, if $0 < v < \infty$, then $f(z)$ is of order ρ_o and type T_o if and only if, $\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T_o = v$.

Now, if $v = 0$ then $f(z)$ is of order at most ρ_o and (7.2.9) gives that if $f(z)$ is of order ρ_o then its type is zero. Hence, if $v = 0$, then $f(z)$ is of growth $\{\rho_o, 0\}$.

Similarly, if $v = \infty$ then $f(z)$ is of order at least ρ_o and (7.2.4) shows that if $f(z)$ is of order ρ_o then $T_o = \infty$. Hence, if $v = \infty$, $f(z)$ is of growth not less that $\{\rho_o, \infty\}$.

The converse also follows in a similar manner. Hence the theorem.

7.3. We observe that a result analogous to that of Theorem 7.1 does not hold always for the lower type t_o. For, consider

$$(7.3.1) \qquad f(z) = \sum_{k=0}^{\infty} \exp (k^{1/4}) z^k + \sum_{k=0}^{\infty} \exp (k^{1/2}) z^{2k}$$

$$= f_1(z) + f_2(z) \quad \text{(say)}.$$

Then it can be easily seen that $f(z)$ is analytic in the unit disc and that $\log \mu(r, f_2) \sim (-8 \log r)^{-1}$ as $r \to 1$. Since, $\log M(r,f) \sim \log M(r,f_2)$, we have, using (1.9.4) and (7.2.1), that $\rho_o = \lambda_o = 1$ and $t_o = 1/8$. But

$$\liminf_{k \to \infty} \frac{1}{k} \left(\frac{\log^+ |a_k|}{2} \right)^2 = 0.$$

However, the following relation always holds.

THEOREM 7.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *belong to the class* U *and have order* $\rho_o (0 < \rho_o < \infty)$ *and lower type* $t_o (0 \le t_o \le \infty)$. *Let* $\{n_k\}_{k=1}^{\infty}$ *be an increasing sequence of natural numbers. Then,*

$$\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} t_o \ge \liminf_{k\to\infty} \lambda_{n_{k-1}} (\log^+ |a_{n_k}|^{1/\lambda_{n_k}})^{\rho_o+1} . \tag{7.3.2}$$

PROOF. Let

$$\liminf_{k\to\infty} \frac{\rho_o^{\rho_o}}{(\rho_o+1)^{\rho_o+1}} \lambda_{n_{k-1}} (\log^+ |a_{n_k}|^{1/\lambda_{n_k}})^{\rho_o+1} = \beta .$$

Then for any ε such that $\beta > \varepsilon > 0$,

$$\log^+ |a_{n_k}| > \frac{(\rho_o+1)\lambda_{n_k}}{\rho_o^{\rho_o/(\rho_o+1)}} \left(\frac{\beta-\varepsilon}{\lambda_{n_{k-1}}}\right)^{1/(\rho_o+1)}$$

for all $k > k_o = k_o(\varepsilon)$. Since the right hand side of the above inequality is positive, we can write for all $k > k_o$

$$\log |a_{n_k}| > \frac{(\rho_o+1)\lambda_{n_k}}{\rho_o^{\rho_o/(\rho_o+1)}} \left(\frac{\beta-\varepsilon}{\lambda_{n_{k-1}}}\right)^{1/(\rho_o+1)} .$$

Let $r_k = \exp\ [-\{(\beta-\varepsilon)\rho_o/\lambda_{n_{k-1}}\}^{1/\rho_o+1}]$, for $k = 2,3,\ldots\ldots$

If $k > k_o$ and $r_k \le r \le r_{k+1}$, then

$$\log M(r) \geq \log |a_{n_k}| + \lambda_{n_k} \log r$$

$$\geq \log |a_{n_k}| + \lambda_{n_k} \log r_k$$

$$> \lambda_{n_k} \left(\frac{\beta-\varepsilon}{\lambda_{n_{k-1}}}\right)^{1/(\rho_o+1)} \rho_o^{-\rho_o/(\rho_o+1)}$$

$$= (\beta-\varepsilon) \frac{\log 1/r_k}{(\log 1/r_{k+1})^{\rho_o+1}}$$

$$\geq (\beta-\varepsilon) (\log 1/r)^{-\rho_o} ,$$

and so,

$$\frac{\log M(r)}{(1-r)^{-\rho_o}} \geq (\beta-\varepsilon) \frac{(\log \frac{1}{r})^{-\rho_o}}{(1-r)^{-\rho_o}} .$$

The above inequality, on proceeding to limits, gives $t \geq \beta$. If $\beta = 0$, the result follows trivially and when $\beta = \infty$ above arguments with an arbitrarily large number in place of $(\beta - \varepsilon)$, give $t = \infty$. Hence the proof of the theorem is complete.

COROLLARY 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *belong to the class* U *and have order* $\rho_o (0 < \rho_o < \infty)$ *and lower type* t_o $(0 \leq t_o \leq \infty)$. *Let* $\{n_k\}_{k=1}^{\infty}$ *be an increasing sequence of natural numbers such that*

$$(7.3.3) \qquad \liminf_{k \to \infty} \frac{\lambda_{n_{k-1}}}{\lambda_{n_k}} = L \qquad (L > 0)$$

then,

$$(7.3.4) \qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} t_o \geq L \liminf_{k \to \infty} \lambda_{n_k} (\log^+ |a_{n_k}|^{1/\lambda_{n_k}})^{\rho_o+1} .$$

COROLLARY 2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ belong to the class U and have order $\rho_o (0 < \rho_o < \infty)$ and lower type $t_o (0 \leq t_o \leq \infty)$. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers such that $\lambda_{n_{k-1}} \sim \lambda_{n_k}$ as $k \to \infty$, then

$$(7.3.5) \qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} t_o \geq \liminf_{k \to \infty} \lambda_{n_k} (\log^+ |a_{n_k}|^{1/\lambda_{n_k}})^{\rho_o+1} .$$

Combining (7.3.5) and Theorem 7.1 we have the following:

COROLLARY 3. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U and have order $\rho_o (0 < \rho_o < \infty)$, type T_o $(0 \leq T_o \leq \infty)$ and lower type $t_o (0 \leq t_o \leq \infty)$. If (i) $\lambda_{k-1} \sim \lambda_k$ as $k \to \infty$ (ii) $\omega = \lim_{k \to \infty} \lambda_k (\log^+ |a_k|^{1/\lambda_k})^{\rho_o+1}$ exists, then $f(z)$ is of perfectly regular growth and $T_o = t_o = \frac{\rho_o^{\rho_o}}{(\rho_o+1)^{\rho_o+1}} \omega$.

THEOREM 7.3. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U and have order $\rho_o (0 < \rho_o < \infty)$ and lower type $t_o (0 \leq t_o \leq \infty)$. Let $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)}$ be a nondecreasing function of k for $k > k_o$ then

$$(7.3.6) \qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} t_o \leq \liminf_{k \to \infty} \lambda_k (\log^+ |a_k|^{1/\lambda_k})^{\rho_o+1} .$$

PROOF. Since, by hypothesis, $\psi(k)$ forms a nondecreasing function of k for $k > k_o$, $\psi(k) > \psi(k-1)$ for infinitely many values of k; for otherwise $\rho_o = 0$. $\psi(k) \to 1$ as $k \to \infty$. When $\psi(k) > \psi(k-1)$, the term $a_k z^{\lambda_k}$ becomes the maximum term and we have

$$\mu(r) = |a_k| r^{\lambda_k}, \nu(r) = \lambda_k \text{ for } \psi(k-1) \leq r < \psi(k).$$

First let $0 < t_o < \infty$. In view of Lemma 7.1 and the definition of the lower type, for given ε such that $t_o > \varepsilon > 0$, we have

$$\log \mu(r) > (t_o-\varepsilon)\ (1-r)^{-\rho_o},$$

for all $r > r_o = r_o(\varepsilon)$. Let $a_{k_1} z^{\lambda_{k_1}}$ and $a_{k_2} z^{\lambda_{k_2}}$ $(k_1 > k_o,\ \psi(k_1-1) > r_o)$ be two consecutive maximum terms so that $k_1 \leq k_2 - 1$, then

$$\log |a_{k_2}| + \lambda_{k_2} \log r > (t_o-\varepsilon)\ (1-r)^{-\rho_o}$$

for all r satisfying $\psi(k_2-1) \leq r < \psi(k_2)$. Let $k_1 \leq k \leq k_2-1$. It is easily seen that

$$\psi(k_1) = \psi(k_1+1) = \dots = \psi(k) = \dots = \psi(k_2-1),$$

and that

$$|a_k| r^{\lambda_k} = |a_{k_2}| r^{\lambda_{k_2}} \text{ for } r = \psi(k).$$

Hence,

$$\log |a_k| + \lambda_k \log \psi(k) = \log |a_{k_2}| + \lambda_{k_2} \log \psi(k)$$

$$> (t_o-\varepsilon)\ (1-\psi(k))^{-\rho_o}.$$

Since $-\log x \geq 1-x$ for all $x > 0$, we have

$$(7.3.7)\quad (\log^{+}|a_k|^{1/\lambda_k})^{\rho_o+1} \geq \frac{[(t_o-\varepsilon)(1-\psi(k))^{-\rho_o} - \lambda_k \log \psi(k)]^{\rho_o+1}}{\lambda_k^{\rho_o+1}}$$

$$= [(1-\psi(k))^{-\rho_o} - (\frac{\lambda_k}{t_o-\varepsilon})\log \psi(k)]^{\rho_o+1} (\frac{t_o-\varepsilon}{\lambda_k})^{\rho_o+1}$$

$$\geq [(1-\psi(k))^{-\rho_o} + (\frac{\lambda_k}{t_o-\varepsilon})(1-\psi(k))]^{\rho_o+1} (\frac{t_o-\varepsilon}{\lambda_k})^{\rho_o+1} .$$

Now the minimum value of the function

$$S(r) = [(1-r)^{-\rho_o} + \frac{\lambda_k}{(t_o-\varepsilon)} (1-r)]$$

is easily seen to be $\{\lambda_k/(t_o - \varepsilon)\rho_o\}^{\rho_o/(1+\rho_o)}(1+\rho_o)$ being attained at $r = 1 - \{\rho_o(t_o-\varepsilon)/\lambda_k\}^{1/(\rho_o+1)} = r_1$(say). Hence,

$$(7.3.8)\qquad S(\psi(k)) \geq S(r_1) \geq (\frac{\lambda_k}{(t_o-\varepsilon)\rho_o})^{\rho_o/(1+\rho_o)}(1+\rho_o).$$

Thus, by (7.3.7) and (7.3.8), we get, for all $k > k_o$

$$(\log^{+} |a_k|^{1/\lambda_k})^{\rho_o+1} \geq \frac{(t_o-\varepsilon)\ (\rho_o+1)^{\rho_o+1}}{\lambda_k\ \rho_o^{\rho_o}} .$$

Multiplying both the sides by λ_k and proceeding to limits, this gives,

$$\liminf_{k\to\infty} \lambda_k (\log^{+} |a_k|^{1/\lambda_k})^{\rho_o+1} \geq \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} t_o,$$

which holds obviously if $t_o = 0$. If $t_o = \infty$, above arguments can be carried with an arbitrarily large number in place of $(t_o - \varepsilon)$ to give

$$\liminf_{k \to \infty} \lambda_k \left(\log^+ |a_k|^{1/\lambda_k}\right)^{\rho_o+1} = \infty.$$

This completes the proof of the theorem.

COROLLARY. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U and have order ρ_o $(0 < \rho_o < \infty)$ and lower type $t_o (0 \leq t_o \leq \infty)$. Let $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)}$ be a nondecreasing function of k for $k > k_o$, then

$$(7.3.9) \qquad L_o \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} t_o \leq \liminf_{k \to \infty} \lambda_{k-1} \left(\log^+ |a_k|^{1/\lambda_k}\right)^{\rho_o+1}$$

where,

$$L_o = \liminf_{k \to \infty} \frac{\lambda_{k-1}}{\lambda_k} \; (L_o > 0).$$

Combining the above theorem and Corollary 2 of Theorem 7.2, we get the following theorem which gives a formula for the lower type t_o for a subclass of function of class U. This formula is analogous to that obtained for the case of type in Theorem 7.1.

THEOREM 7.4. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be analytic in the unit disc with order $\rho_o (0 < \rho_o < \infty)$ and lower type t_o $(0 \leq t_o \leq \infty)$. If

(i) $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)}$ forms a nondecreasing function of k for $k > k_o$ and (ii) $\lambda_{k+1} \sim \lambda_k$ as $k \to \infty$, then

$$(7.3.10)\qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\, t_o = \liminf_{k\to\infty} \lambda_k \left(\log^+ |a_k|^{1/\lambda_k}\right)^{\rho_o+1} .$$

Our next theorem gives a coefficient characterization of the lower type, which holds for a wider subclass of functions of the class U .

THEOREM 7.5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ belong to the class U and have order $\rho_o (0 < \rho_o < \infty)$ and lower type $t_o (0 \le t_o \le \infty)$. Let $\{\lambda_{n_k}\}_{k=1}^{\infty}$ be the sequence of principal indices of $f(z)$ such that $\lambda_{n_{k-1}} \sim \lambda_{n_k}$ as $k \to \infty$, then

$$(7.3.11)\qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\, t_o = \max_{\{m_k\}} \left[\liminf_{k\to\infty} \lambda_{m_{k-1}} \left(\log^+ |a_{m_k}|^{1/\lambda_{m_k}}\right)^{\rho_o+1}\right]$$

where maximum in (7.3.11) is taken over all increasing sequences $\{m_k\}_{k=1}^{\infty}$ of natural numbers.

PROOF. Consider the function $g(z) = \sum_{k=0}^{\infty} a_{n_k} z^{\lambda_{n_k}}$, $\{\lambda_{n_k}\}_{k=0}^{\infty}$ being the principal indices of $f(z)$. It is easily seen that $g(z) \in U$ and $f(z)$ and $g(z)$ have same maximum term for every z in the unit disc. Hence, by (1.9.4) and Lemma 7.1, $g(z)$ is of order ρ_o and lower type t_o. Since $\psi(n_k) \equiv |a_{n_k}/a_{n_{k+1}}|^{1/(\lambda_{n_{k+1}} - \lambda_{n_k})}$ are jump points of the rank $\nu(r)$ of $f(z)$, they form an increasing sequence . Thus $g(z)$ satisfies the

hypothesis of Theorem 7.4, hence by (7.3.10),

$$(7.3.12)\qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\, t_o = \liminf_{k\to\infty} \lambda_{n_{k-1}} (\log^+ |a_{n_k}|^{1/\lambda_{n_k}})^{\rho_o+1}$$

But, by Theorem 7.2,

$$(7.3.13)\qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\, t_o \geq \max_{\{m_k\}} \liminf_{k\to\infty} \lambda_{m_{k-1}} (\log^+ |a_{m_k}|^{1/\lambda_{m_k}})^{\rho_o+1}$$

Combining (7.3.12) and (7.3.13), we get (7.3.11) and the proof of the theorem is complete.

7.4. In this section we find coefficient characterizations of the growth number μ_o and lower growth number δ_o for a subclass of functions of U. We also obtain a decomposition theorem which is analogous to a theorem of L.R. Sons [75, Theorem 2].

THEOREM 7.6. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U and have order $\rho_o (0 < \rho_o < \infty)$, growth number $\mu_o (0 \leq \mu_o \leq \infty)$ and lower growth number δ_o $(0 \leq \delta_o \leq \infty)$. If $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ forms a strictly increasing function of k for $k > k_o$, then

$$(7.4.1)\qquad \mu_o = \limsup_{k\to\infty} \lambda_k \left(\frac{1}{\lambda_k-\lambda_{k-1}} \log |a_k/a_{k-1}|\right)^{\rho_o+1}$$

and

$$(7.4.2) \qquad \delta_o = \liminf_{k \to \infty} \lambda_{k-1} \left(\frac{1}{\lambda_k - \lambda_{k-1}} \log |a_k/a_{k-1}|\right)^{\rho_o+1}.$$

PROOF. First, let $\mu_o < \infty$. By (7.1.2), for any $\varepsilon > 0$ and for all r such that $1 > r > r_o = r_o(\varepsilon)$, we have

$$\nu(r) < (\mu_o+\varepsilon)\, r(1-r)^{-\rho_o-1}.$$

Since $\psi(k)$ forms a strictly increasing function of k for $k > k_o$

$$\nu(r) = \lambda_k \text{ for } \psi(k-1) \leq r < \psi(k).$$

Hence,

$$\lambda_k < (\mu_o+\varepsilon)\, r\, (1-r)^{-\rho_o-1}.$$

Since the above inequality holds for all $r \geq \psi(k-1)$, we have

$$\mu_o + \varepsilon > \frac{\lambda_k\, (1-\psi(k-1))^{\rho_o+1}}{\psi(k-1)} \simeq \frac{\lambda_k \left(\log \frac{1}{\psi(k-1)}\right)^{\rho_o+1}}{\psi(k-1)}.$$

Since, $\psi(k) \to 1$ as $k \to \infty$ we have, on proceeding to limit,

$$(7.4.3) \qquad \mu_o \geq \limsup_{k \to \infty} \lambda_k \left(\frac{1}{\lambda_k - \lambda_{k-1}} \log |a_k/a_{k-1}|\right)^{\rho_o+1}.$$

This inequality holds obviously if $\mu_o = \infty$.

Next assume that $\mu_o > 0$. Then for every ε such that $\mu_o > \varepsilon > 0$, there exists a sequence $\{r_p\}$ tending to one such that

$$(7.4.4) \qquad \nu(r_p) > (\mu_o-\varepsilon)\, r_p(1-r_p)^{-\rho_o-1} \quad \text{for } p = 1,2,3,\ldots .$$

Since $\psi(k)$ is a strictly increasing function and tends to 1 as $k \to \infty$, for every r_p we can find an integer k_p such that $\psi(k_p-1) \le r_p < \psi(k_p)$, and so we have, $\nu(r_p) = \lambda_{k_p}$. Thus (7.4.4) gives,

$$\mu_o - \varepsilon < \frac{\lambda_{k_p}(1-\psi(k_p-1))^{\rho_o+1}}{\psi(k_p-1)}$$

$$\simeq \lambda_{k_p}\left(\log \frac{1}{\psi(k_p-1)}\right)^{\rho_o+1},$$

which gives, on proceeding to limits,

$$(7.4.5) \qquad \mu_o \le \limsup_{k \to \infty} \lambda_k \left(\frac{1}{\lambda_k-\lambda_{k-1}} \log |a_k/a_{k-1}|\right)^{\rho_o+1}.$$

This inequality is obviously true if $\mu_o = 0$.

(7.4.3) and (7.4.5) give (7.4.1). (7.4.2) can be proved in a similar manner. Hence the theorem.

THEOREM 7.7. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class $\mathcal{U}$, have order $\rho_o(0 < \rho_o < \infty)$, growth number μ_o and lower growth number δ_o, then

$$(7.4.6) \qquad \delta_o \le \mu_o \liminf_{k \to \infty} \frac{\lambda_k}{\lambda_{k+1}}.$$

PROOF. Let $\alpha = \liminf_{k \to \infty} \frac{\lambda_k}{\lambda_{k+1}}$. If $\beta > \alpha$, there exists a sequence $\{c(k)\}$ such that $\lambda_{c(k)} < \beta\, \lambda_{c(k)+1}$. Let r_t be a value of r at which $\nu(r)$ jumps from a value less than or equal to $\lambda_{c(t)}$ to a value greater than or equal to $\lambda_{c(t)+1}$. Now, since

$$\nu(r_t - 0) \leq \lambda_{c(t)} < \beta\, \lambda_{c(t)+1} \leq \beta\, \nu(r_t+0),$$

therefore,

$$\delta_o \leq \limsup_{t \to \infty} \frac{\nu(r_t-0)}{r_t(1-r_t)^{-\rho_o-1}} \leq \beta \limsup_{t \to \infty} \frac{\nu(r_t+0)}{r_t(1-r_t)^{-\rho_o-1}} \leq \mu_o\, \beta.$$

Since the last inequality holds for all $\beta > \alpha$, hence $\delta_o \leq \alpha\, \mu_o$, which proves (7.4.6).

REMARK. If $\delta_o = \mu_o$ then $\lambda_k \sim \lambda_{k+1}$ as $k \to \infty$.

THEOREM 7.8. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *belong to the class* U *and have order* $\rho_o (0 < \rho_o < \infty)$, *type* T_o *and lower type* t_o $(0 < t_o \leq T_o < \infty)$ *and let* μ *be such that* $t_o < \mu < T_o$. *Then*,

$$f(z) = g(z) + h(z)$$

where $g(z)$ *is of growth* $\{\rho_o, \mu\}$ *and* $h(z) = \sum_{k=0}^{\infty} b_k z^{m_k}$ $(b_k \neq 0$ *for all* $k)$ *satisfies*

$$t_o \geq \mu \liminf_{k \to \infty} \left(\frac{m_k}{m_{k+1}}\right)^{\rho_o/(1+\rho_o)} . \tag{7.4.7}$$

PROOF. Let $g(z) = \sum_{k=0}^{\infty} c_k z^k$, where

$$c_k = a_k \quad \text{if} \quad \log^+ |a_k| < B\, k^{\rho_o/(1+\rho_o)}$$
$$= 0 \quad \text{otherwise,}$$

where $B = \mu^{1/(\rho_o+1)} (\rho_o+1)\rho_o^{-\rho_o/(1+\rho_o)}$. Then $g(z) \in U$ and is of growth $\{\rho_o, \mu\}$. Set $h(z) = f(z) - g(z) = \sum_{k=0}^{\infty} b_k z^{m_k}$, and let $A_k = |b_k|$, then

$$\log^+ A_k \geq B\, m_k^{\rho_o/(1+\rho_o)} .$$

Let,

$$-\log r_k = \frac{\rho_o}{\rho_o+1} \frac{B}{} m_k^{-1/(\rho_o+1)} \quad \text{for } k = 1,2,\ldots .$$

If $r_k \leq r \leq r_{k+1}$, then

$$\begin{aligned} \log M(r,f) &\geq \log A_k + m_k \log r \\ &\geq \log A_k + m_k \log r_k \\ &> B\, m_k^{\rho_o/(1+\rho_o)} \frac{1}{1+\rho_o} , \end{aligned}$$

which gives,

$$\frac{\log M(r,f)}{(1-r)^{-\rho_o}} > \frac{B\, m_k^{\rho_o/(1+\rho_o)} (1/(1+\rho_o))}{(-\log r_{k+1})^{-\rho_o}} = \mu\, (m_k/m_{k+1})^{\rho_o/(1+\rho_o)} .$$

Now, on proceeding to limits,

$$t_o \geq \mu \liminf_{k\to\infty} \left(\frac{m_k}{m_{k+1}}\right)^{\rho_o/(1+\rho_o)} .$$

REMARK. The above theorem is an analogue of L.R. Sons' result [75, Theorem 1] concerning order and lower order.

7.5. In this section for a function $f(z) \in U$, we obtain some relations involving its type, lower type and ratio of the consecutive coefficients in its Taylor series expansion.

THEOREM 7.9. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ belong to the class U, have order $\rho_0 (0 < \rho_0 < \infty)$, type T_0 and lower type $t_0 (0 \leq t_0 \leq T_0 < \infty)$ then,

$$\frac{1}{\rho_0}\left(\frac{\rho_0+\alpha}{\rho_0+1}\right)^{\rho_0+1} R_0 \leq t_0 \leq T_0 \leq Q_0/\rho_0 \tag{7.5.1}$$

where,

$$R_0 = \liminf_{k\to\infty} \lambda_{k-1}\left(\frac{1}{\lambda_k-\lambda_{k-1}} \log^+ |a_k/a_{k-1}|\right)^{\rho_0+1}$$

and

$$Q_0 = \limsup_{k\to\infty} \lambda_k \left(\frac{1}{\lambda_k-\lambda_{k-1}} \log^+ |a_k/a_{k-1}|\right)^{\rho_0+1},$$

and

$$\alpha = \liminf_{k\to\infty} \frac{\lambda_{k-1}}{\lambda_k}.$$

PROOF. First assume that $0 < R_0 < \infty$. For any ε such that $R_0 > \varepsilon > 0$ we have for all $m > N = N(\varepsilon)$

$$\log^+ |a_m/a_{m-1}| > (R_0-\varepsilon)^{1/(\rho_0+1)} \lambda_{m-1}^{-1/(\rho_0+1)} (\lambda_m - \lambda_{m-1}).$$

Writing the above inequality for $m = N+1,\dots,k$ and adding all such inequalities, we get

$$\log^+|a_k| > (R_o-\varepsilon)^{1/(\rho_o+1)}\{\lambda_k\lambda_{k-1}^{-1/(\rho_o+1)} - \sum_{m=N+1}^{k-1}\lambda_m(\lambda_m^{-1/(\rho_o+1)}-\lambda_{m-1}^{-1/(\rho_o+1)}) - \lambda_N^{-\rho_o/(\rho_o+1)}\} + \log|a_N|$$

$$= (R_o-\varepsilon)^{1/(\rho_o+1)}\{\lambda_k\lambda_{k-1}^{-1/(\rho_o+1)} - \int_{\lambda_N}^{\lambda_{k-1}} n(t)\,d(t^{-1/(\rho_o+1)}) - \lambda_N^{\rho_o/(\rho_o+1)}\} + \log|a_N|$$

where $n(t) = \lambda_m$ for $\lambda_{m-1} \le t < \lambda_m$. Therefore,

$$\log^+|a_k| > (R_o-\varepsilon)^{1/(\rho_o+1)}\{\lambda_k\lambda_{k-1}^{-1/(\rho_o+1)} + \frac{1}{\rho_o+1}\int_{\lambda_N}^{\lambda_{k-1}} \frac{n(t)}{t} \times t^{-1/(\rho_o+1)}\,dt - \lambda_N^{\rho_o/(\rho_o+1)}\} + \log|a_N|$$

$$> (R_o-\varepsilon)^{1/(\rho_o+1)}\{\lambda_k\lambda_{k-1}^{-1/(\rho_o+1)} + \frac{1}{\rho_o}\lambda_{k-1}^{\rho_o/(1+\rho_o)} - \frac{\rho_o+1}{\rho_o}\lambda_N^{\rho_o/(\rho_o+1)}\} + \log|a_N|,$$

which gives,

$$\lambda_{k-1}(\log^+|a_k|^{1/\lambda_k})^{\rho_o+1} > (R_o-\varepsilon)\{1 + \frac{1}{\rho_o}\frac{\lambda_{k-1}}{\lambda_k} - o(1)\}^{\rho_o+1} + o(1).$$

Thus, on proceeding to limits, and using (7.3.2), we get

$$\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\, t_o \ge (R_o-\varepsilon)\left(\frac{\rho_o+\alpha}{\rho_o}\right)^{\rho_o+1}.$$

Since ε is arbitrary, we have,

$$t_o \ge R_o\left(\frac{\rho_o+\alpha}{\rho_o+1}\right)^{\rho_o+1}\frac{1}{\rho_o}.$$

This inequality is obviously true if $R_o = 0$. If $R_o = \infty$, above arguments with an arbitrarily large number in place of $(R_o - \varepsilon)$ give $t_o = \infty$.

Next let $Q_o < \infty$. For any $\varepsilon > 0$ and all $m > N = N(\varepsilon)$, we have

$$\log^+ |a_m/a_{m-1}| < (Q_o+\varepsilon)^{1/(\rho_o+1)} \lambda_m^{-1/(\rho_o+1)} (\lambda_m - \lambda_{m-1}).$$

Writing the above inequality for $m = N+1, N+2, \ldots, k$ and adding all such inequalities, we get

$$\log^+|a_k| < (Q_o+\varepsilon)^{1/(\rho_o+1)} \{\lambda_k^{\rho_o/(\rho_o+1)} - \sum_{m=N+1}^{k-1} \lambda_m(\lambda_{m+1}^{-1/(\rho_o+1)} - \lambda_m^{-1/(\rho_o+1)}) - \lambda_N\lambda_{N+1}^{-1/(\rho_o+1)}\}+\log|a_N|,$$

$$= (Q_o+\varepsilon)^{1/(\rho_o+1)} \{\lambda_k^{\rho_o/(\rho_o+1)} - \int_{\lambda_{N+1}}^{\lambda_k} n(t)d(t^{-1/(\rho_o+1)}) - \lambda_N\lambda_{N+1}^{-1/(\rho_o+1)}\}+\log|a_N|$$

where $n(t) = \lambda_m$ for $\lambda_m \le t < \lambda_{m+1}$. This gives,

$$\log^+|a_k| < (Q_o+\varepsilon)^{1/(\rho_o+1)} \{\lambda_k^{\rho_o/(\rho_o+1)} + \frac{1}{\rho_o+1}\int_{\lambda_{N+1}}^{\lambda_k} t^{-1/(\rho_o+1)} dt - \lambda_N\lambda_{N+1}^{-1/(\rho_o+1)}\}+\log|a_N|$$

$$= (Q_o+\varepsilon)^{1/(\rho_o+1)} \left(\frac{\rho_o+1}{\rho_o}\right)\lambda_k^{\rho_o/(\rho_o+1)} \{1-o(1)\} + o(1).$$

Therefore,

$$\lambda_k(\log^+ |a_k|^{1/\lambda_k})^{\rho_o+1} < (Q_o+\varepsilon) \left(\frac{\rho_o+1}{\rho_o}\right)^{\rho_o+1}.$$

On proceeding to limits and using Theorem 7.1, we get

$$\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T_o \leq Q_o \left(\frac{\rho_o+1}{\rho_o}\right)^{\rho_o+1},$$

and hence, $T_o \leq Q_o/\rho_o$. This inequality is obviously true if $Q_o = \infty$. This completes the proof of the theorem.

REMARK. From (7.5.1) it is evident that if $\lambda_{k-1} \sim \lambda_k$ and if

$$\omega^* \equiv \lim_{k\to\infty} \lambda_k \left(\frac{1}{\lambda_k - \lambda_{k-1}} \log^+ |a_k/a_{k-1}| \right)^{\rho_o+1}$$

exists then $f(z)$ is of perfectly regular growth, i.e., $T_o = t_o = \omega^*/\rho_o$. But the converse need not be true always. Thus, for

$$f(z) = \sum_{k=0}^{\infty} \exp\left((2k+1)^{1/2}\right) z^{2k+1} + \sum_{k=0}^{\infty} \exp\left(k^{1/2}\right) z^k$$

we have $\rho_o = \lambda_o = 1$, $T_o = t_o = 1/4$ while $R_o = 0$ and $Q_o = \infty$.

THEOREM 7.10. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *belong to the class* U, *have order* $\rho_o (0 < \rho_o < \infty)$ *and type* T_o $(0 \leq T_o \leq \infty)$. *Further, let* $\psi(k) \equiv |a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ *be a nondecreasing function of* k *for* $k > k_o$, *then*

(7.5.2)
$$\rho_o T_o \leq Q_o \leq \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T_o.$$

where Q_o is defined as in Theorem 7.9.

PROOF. Since $\psi(k)$ is nondecreasing, we note that $\log^+$ in the definition of Q_o can be replaced by log. Since $\rho_o T_o \le Q_o$ is contained in (7.5.1), we have only to prove $Q_o \le \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T_o$. This inequality is obviously true when $T_o = \infty$, hence assume that $T_o < \infty$. By Theorem 7.1 for any $\varepsilon > 0$ and for all $k > N = N(\varepsilon)$, we get

$$(7.5.3) \qquad \log |a_k| < \frac{(\rho_o+1)(T+\varepsilon)^{1/(\rho_o+1)}}{\rho_o^{\rho_o/(\rho_o+1)}} \lambda_k^{\rho_o/(\rho_o+1)} .$$

Since $\psi(k)$ is nondecreasing for $k > k_o$, we have

$$\log |a_k| = \log |a_N| + \sum_{m=N+1}^{k} (\lambda_m - \lambda_{m-1}) \log \frac{1}{\psi(m-1)}$$

$$\ge \log |a_N| + (\lambda_k - \lambda_N) \log \frac{1}{\psi(k-1)} .$$

Thus, by (7.5.3),

$$\frac{(\rho_o+1)(T+\varepsilon)^{1/(\rho_o+1)}}{\rho_o^{\rho_o/(\rho_o+1)}} \lambda_k^{\rho_o/(\rho_o+1)} \ge \log |a_N| + \lambda_k\left(1 - \frac{\lambda_N}{\lambda_k}\right)\log \frac{1}{\psi(k-1)} ,$$

or,

$$\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} (T+\varepsilon) \ge o(1) + \lambda_k (1-o(1)) \left[\log \frac{1}{\psi(k-1)}\right]^{\rho_o+1} .$$

Substituting the value of $\psi(k-1)$ and proceeding to limits, we get,

$$\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T \geq \limsup_{k\to\infty} \lambda_k \left[\frac{1}{\lambda_k-\lambda_{k-1}} \log |a_k/a_{k-1}|\right]^{\rho_o+1} .$$

This completes the proof of the theorem.

REMARK. Let $r_k = \exp\{(\frac{\rho_o T_o}{k})^{1/(\rho_o+1)}\}$ $(0 < \rho_o, T_o < \infty)$ and let $f(z) = 1 + \sum_{k=1}^{\infty} (r_1 \ldots r_k) z^k$. Then $f(z) \in U$ and $|a_{k-1}/a_k| = 1/r_k$ is an increasing function of k. Further, it can be easily seen that order of $f(z)$ is ρ_o, type of $f(z)$ is T_o and $Q_o = \rho_o T_o$. Hence the inequality $\rho_o T_o \leq Q_o$ is best possible. The example (6.3.5) shows that the inequality $Q \leq \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T_o$ is also best possible. For the function given in this example $\rho_o = 1$ and $\zeta_o = 0$. Let,

$$\phi(k) = k (\log r_k)^2 .$$

Then, $\phi(n_k^2) \to 0$ as $k \to \infty$ and $\phi(n_{k+1}) \to 1$ as $k \to \infty$. Hence it can be easily seen that $Q_o = 1$. Now, let

$$\alpha(k) = \frac{(\log |a_k|)^2}{k} .$$

Then $\alpha(n_k^2) \to 0$ as $k \to \infty$ while $\alpha(n_{k+1}) \to 1$ as $k \to \infty$. Hence by Theorem 7.1 the type of the function $f(z)$ is 1/4 and consequently

$$Q_o = \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T_o$$ holds.

CHAPTER 8

D-PROXIMATE ORDER AND MAXIMUM TERM OF FUNCTIONS ANALYTIC IN THE UNIT DISC

8.1. In Chapters 6 and 7, we saw that the growth of a function $f(z)$ of the class U can be studied in terms of its order ρ_o and type T_o, but it is evident that these concepts are inadequate to compare the growth of those functions of the class U which are of the same nonzero finite order and of infinite type. Hence, for a refinement of the above growth scale, one may compare $\log M(r)$ with the function

$$(1-r)^{-\rho_o}\left(\log\frac{1}{1-r}\right)^{\alpha_1}\left(\log^{[2]}\frac{1}{1-r}\right)^{\alpha_2}\cdots\left(\log^{[p]}\frac{1}{1-r}\right)^{\alpha_p},\quad (0<\rho_o,\alpha_1,\ldots,\alpha_p<\infty),$$

where $\log^{[0]} x = x$ and $\log^{[j]} x = \log(\log^{[j-1]} x)$, $1 \le j \le p$. Thus a generalized order of $f(z)$ could be defined as the system of numbers $(\rho_o, \alpha_1, \ldots, \alpha_p)$ such that

$$T(\rho_o,\alpha_1,\ldots,\alpha_p) = \limsup_{r\to 1}\frac{\log M(r)}{(1-r)^{-\rho_o}\left(\log\frac{1}{1-r}\right)^{\alpha_1}\left(\log^{[2]}\frac{1}{1-r}\right)^{\alpha_2}\cdots\left(\log^{[p]}\frac{1}{1-r}\right)^{\alpha_p}}$$

is nonzero finite. The generalized order $(\rho_o, \alpha_1, \ldots, \alpha_p)$ of a function $f_1(z) \in U$ is said to be greater than the generalized order $(\rho_o', \alpha_1', \ldots, \alpha_p')$ of a function $f_2(z) \in U$, if either $\rho_o > \rho_o'$ or $\rho_o = \rho_o'$, $\alpha_1 > \alpha_1'$ or $\rho_o = \rho_o'$, $\alpha_j = \alpha_j'$ and $\alpha_{j+1} > \alpha_{j+1}'$, $1 \le j < p$.

A study of these growth constants can be done in a manner analogous to that pursued in previous chapters but we will not deviate in that direction since we wish to study the growth of $f(z)$ in a more general manner.

In fact, the grwoth scale mentioned above can be further refined by introducing a certain class of 'slowly increasing' functions $L(r)$ on the open interval $(0,1)$ and then comparing the growth of $\log M(r)$ with that of $(1-r)^{-\rho_0} L(r)$. For this purpose we introduce the concept of a D-proximate order $\rho_0(r)$ as follows:

DEFINITION 8.1. *A real valued function $\rho_0(r)$ defined on $(0,1)$ is called a D-proximate order if it possesses the following properties:*

(8.1.1) $\rho_0(r)$ *is a positive, continuous and piecewise differentiable function for all r such that $0 < r_0 < r < 1$;*

(8.1.2) $$\lim_{r \to 1} \rho_0(r) = \rho_0 \quad (0 < \rho_0 < \infty);$$

(8.1.3) $$\lim_{r \to 1} - \rho_0'(r)\,(1-r)\log(1-r) = 0,$$ *where $\rho_0'(r)$ is either the right or the left derivative of $\rho_0(r)$ where these are different.*

We define the generalized type T^* and lower generalized type t^* of $f(z)$ with respect to a given D-proximate order $\rho_0(r)$ as

(8.1.4) $$\begin{matrix} T^* \\ t^* \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r)}{(1-r)^{-\rho_0(r)}} \quad (0 \le t^* \le T^* \le \infty).$$

DEFINITION 8.2. *A D-proximate order $\rho_0(r)$ is called a D-proximate order of $f(z) \in U$ if T^* is nonzero and finite.*

We start by defining a slowly increasing function on $(0,1)$.

DEFINITION 8.4. *A real valued function* $L(r)$ $(0 < r < 1)$ *is said to be slowly increasing in* $(0,1)$ *if for every* k *such that* $1 < k < \infty$,

$$(8.2.1) \qquad \lim_{r \to 1} \frac{L(r + \frac{1}{k}(1-r))}{L(r)} = 1 .$$

We now state a few elementary lemmas concerning a D-proximate order. These lemmas follow in a straightforward manner on the same lines as given in Levin [44, pp. 32-34] for the corresponding lemmas on proximate order of entire functions. Hence we omit their proof.

LEMMA 8.1. *Let* $\rho_0(r)$ *be a D-proximate order, then* $L(r) \equiv (1-r)^{-\rho_0(r)+\rho}$ *is a slowly increasing function of* r *for* $0 < r < 1$. *Further, for every* $\varepsilon > 0$, *the inequality*

$$(8.2.2) \qquad (1-\varepsilon)\left(\frac{k}{k-1}\right)^{\rho_0}(1-r)^{-\rho_0(r)} < \left\{\left(\frac{k}{k-1}\right)(1-r)^{-1}\right\}^{\rho_0(r+\frac{1}{k}(1-r))} <$$

$$< (1+\varepsilon)\left(\frac{k}{k-1}\right)^{\rho_0}(1-r)^{-\rho_0(r)}$$

holds for all r *such that* $0 < r_0 = r_0(\varepsilon) < r < 1$ *and for all* $k > 1$.

LEMMA 8.2. *Let* $\rho_0(r)$ *be a D-proximate order, then the function* $(1-r)^{-\rho_0(r)}$ *is monotonically increasing function of* r *for* $r > r_0$, $(0 < r_0 < r < 1)$.

LEMMA 8.3. *Let* $\rho(r)$ *be a D-proximate order then for* $\rho_0 - 1 > \xi \geq -1$ *and for* $0 < r_0 < \alpha < r < 1$, *we have*

$$(8.2.3)\quad \int_{\alpha}^{r}(1-t)^{-\rho_0(t)+\xi}\,dt = \frac{1}{\rho_0-\xi-1}(1-r)^{-\rho_0(r)+\xi+1} + o((1-r)^{-\rho_0(r)+\xi+1}).$$

Since, by lemma 8.2, $(1-r)^{-\rho_0(r)}$ is a monotonically increasing function of r for $0 < r_0 < r < 1$, a single-valued real function $\chi(t)$ of t can be defined for $t > t_0$ such that

$$(8.2.4)\qquad t = (1-r)^{-\rho_0(r)-1} \text{ if and only if } (1-r)^{-1} = \chi(t).$$

Then we have

LEMMA 8.4. *Let $\rho_0(r)$ be a D-proximate order and let $\chi(t)$ be defined as in (8.2.4). Then*

$$(8.2.5)\qquad \lim_{t\to\infty}\frac{d(\log\chi(t))}{d(\log t)} = \frac{1}{\rho_0+1}$$

and for every η such that $0 < \eta < \infty$

$$(8.2.6)\qquad \lim_{t\to\infty}\frac{\chi(\eta t)}{\chi(t)} = \eta^{1/(\rho_0+1)}.$$

Next we show that every function of the class $\mathcal{U}$ which is of nonzero finite order has a D-proximate order. Since the lines of proof are analogous to those adopted by Levin [44, pp. 35-39] for proving the existence of a proximate order for entire functions, we only give a brief outline of the proof.

THEOREM 8.1. *Let* $f(z)$ *belong to the class* U *and have the order* $\rho_o (0 < \rho_o < \infty)$. *Then for every* T^* $(0 < T^* < \infty)$, *there exists a proximate order satisfying* (8.1.1) *to* (8.1.3) *such that* $f(z)$ *is of generalized type* T^* *with respect to the proximate order* $\rho_o(r)$.

PROOF. Let $p(r) = \frac{1}{T^*} (1-r)^{-\rho_o} \log M(r)$. If we put $x = -\log (1-r)$ and $p_1(x) = \log p(1-e^{-x})$ then $\limsup_{x \to \infty}(p_1(x)/x) = 0$. First suppose that $\limsup_{x \to \infty} p_1(x) = \infty$. The boundary curve $y = q(x)$ of the smallest convex domain containing the curve $y = p_1(x)$ and the positive ray of the x-axis has the following properties (a) $y = q(x)$ is a concave function and $\lim_{x \to \infty} (q(x)/x) = 0$ (b) $p_1(x) \leq q(x)$ (c) $y = q(x)$ has infinitely many extremal points where $p_1(x) = q(x)$. It follows from (b) that

$$\log M(r) \leq T^* (1-r)^{-\rho_o - \frac{q(-\log (1-r))}{-\log (1-r)}} .$$

If we put $\rho_o(r) = \rho_o + q(-\log (1-r))/-\log (1-r)$, then it is easily seen that $\rho_o(r)$ satisfies all the properties of a D-proximate order for $f(z)$. In the general case, we draw the line segments exactly in the same way as done in Levin [44, pp. 35-39] and construct the polygonal function $\bar{q}_1(x)$ with the property that $\lim_{x \to \infty} (\bar{q}_1(x)/x) = 0$. The function $q_1(x) = -\bar{q}_1(x)$ is seen to have the following properties (a') $\lim_{x \to \infty} (q_1(x)/x) = 0$ (b') $\lim_{x \to \infty} q_1'(x) = 0$ (c') $\lim_{x \to \infty} [p_1(x) + q_1(x)] = \infty$. Now constructing a concave majorant $q_2(x)$ for $p_1(x) + q_1(x)$ and putting $q(x) = q_2(x) - q_1(x)$ it can be proved that $\rho_o(r) = \rho_o + q(-\log(1-r))/-\log(1-r)$ is a D-proximate order for $f(z)$. This completes the proof of the theorem.

REMARK. Analogous to the case of entire functions, the concepts of a lower D-proximate order, D-proximate type, lower D-proximate type etc. can be introduced and their existence can be proved as above. However, we shall not go into the details of these.

8.3. In this section we obtain coefficient characterizations of generalized type T^* and lower generalized type t^* of a function $f(z)$ in U. The coefficient characterization for the lower type holds for those functions in U which satisfy (6.2.8).

First we state a lemma.

LEMMA 8.5. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belong to the class U and have a D-proximate order $\rho_o(r)$. Let T^* and t^* be the generalized type and lower generalized type of $f(z)$ with respect to the D-proximate order $\rho_o(r)$. Then

$$\begin{matrix} T^* \\ t^* \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \mu(r)}{(1-r)^{-\rho_o(r)}} . \tag{8.3.1}$$

The lemma follows on the lines similar (see e.g. [40]) to those of lemma 7.1. Hence we omit the proof.

THEOREM 8.2. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belong to the class U, have the D-proximate order $\rho_o(r)$ and order $\rho_o (0 < \rho_o < \infty)$. Then generalized type T^* of $f(z)$ with respect to the D-proximate order $\rho_o(r)$ is given by

$$\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T^* = \limsup_{k \to \infty} \left\{\frac{\chi(k) \log^+ |a_k|}{k}\right\}^{\rho_o+1} \tag{8.3.2}$$

where $\chi(k)$ *is defined by* (8.2.4).

PROOF. From (8.1.4), for every $\varepsilon > 0$ and for all $r > r_0 (0 < r_0 = r_0(\varepsilon) < r < 1$

$$\log M(r) < (T^* + \varepsilon)(1 - r)^{-\rho_0(r)}.$$

Using Cauchy's estimate, this gives for all r such that $0 < r_0 < r < 1$,

$$\log^+ |a_k| < (T^*+\varepsilon)(1-r)^{-\rho_0(r)} - k \log r. \tag{8.3.3}$$

Now choose $(1-r)^{-\rho_0(r)-1} = \{k/(T^*+\varepsilon)\rho_0\}$ and let $H(k) = (1 - \frac{1}{\chi(k/(T^*+\varepsilon)\rho_0)})$. $H(k) \to 1$ as $k \to \infty$. Put $P(k) = 1/\{\rho_0(H(k)) + 1\}$. Now for all $k > k_0$ we get, by (8.3.3),

$$\log^+ |a_k| < \frac{(T^*+\varepsilon)^{P(k)} k^{1-P(k)}}{\rho_0^{1-P(k)}} - k \log H(k).$$

Therefore,

$$\frac{\chi(k) \log^+ |a_k|}{k} < \frac{(T^*+\varepsilon)^{P(k)}}{\rho_0^{1-P(k)}} \frac{\chi(k)}{k^{P(k)}} [1 - \frac{\rho_0 k^{P(k)} \log H(k)}{\{(T^*+\varepsilon)\rho_0\}^{P(k)}}]. \tag{8.3.4}$$

Since,

$$\lim_{k \to \infty} \frac{\chi(k)}{k^{P(k)}} = 1 \text{ and } \lim_{k \to \infty} \frac{k^{P(k)} \log H(k)}{\{(T^*+\varepsilon)\rho_0\}^{P(k)}} = -1,$$

we have, by (8.1.2) and (8.3.4), that

$$(8.3.5) \qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T^* \geq \limsup_{k\to\infty} \left\{\frac{\chi(k)\log^+|a_k|}{k}\right\}^{\rho_o+1} .$$

Next, let α be defined by the equation

$$\limsup_{k\to\infty} \left\{\frac{\chi(k)\log^+|a_k|}{k}\right\}^{\rho_o+1} = \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} \alpha .$$

Then for every $\beta > \alpha$ and for all $r > r_o (0 < r_o < r < 1)$, we have

$$|a_k| r^k < \exp\left\{\frac{k(\rho_o+1)\beta^{1/(\rho_o+1)}}{\rho_o^{\rho_o/(\rho_o+1)}\chi(k)} + k\log r\right\} .$$

Using Lemma 8.4, we have asymptotically,

$$|a_k| r^k < \exp\left\{\frac{k(\rho_o+1)}{\rho_o\,\chi(k/\beta\rho_o)} - k(1-r)\right\} .$$

Hence, for r sufficiently close to 1,

$$\log \mu(r) < \max_{k\geq 0} \left\{\frac{k(\rho_o+1)}{\rho_o\,\chi(k/\beta\rho_o)} - k(1-r)\right\}.$$

Using (8.2.5), it can be easily seen that the maximum value on the right hand side is attained for

$$k = [\beta\,\rho_o(1-r)^{-\rho_o(r)-1}].$$

Thus, for r sufficiently close to 1, we get,

$$\frac{\log \mu(r)}{(1-r)^{-\rho_o(r)}} < \beta.$$

Proceeding to limit and using Lemma 8.5, this given $T^* \leq \beta$. Since the last inequality holds for all $\beta > \alpha$, we have $T^* \leq \alpha$. This and (8.3.5) together prove (8.3.2).

Taking $\rho_o(r) = \rho_o$, we get $\chi(t) = t^{1/(\rho_o+1)}$. Hence we have the following corollary which gives a formula for the type T_o of $f(z) \in \mathcal{U}$.

COROLLARY. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belong to the class $\mathcal{U}$, have the order $\rho_o (0 < \rho_o < \infty)$ and type T_o $(0 < T_o < \infty)$, then*

$$\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} T_o = \limsup_{k \to \infty} k(\log^+ |a_k|^{1/k})^{\rho_o+1} .$$

It is to be observed that this corollary also follows from Theorem 7.1 by taking $\lambda_k = k$.

THEOREM 8.3. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belong to the class $\mathcal{U}$, have the D-proximate order $\rho_o(r)$ and order $\rho_o (0 < \rho_o < \infty)$ such that $\psi(k) \equiv |a_k/a_{k+1}|$ forms a nondecreasing function of k for $k > k_o$. Then generalized lower type t^* of $f(z)$ is given by*

$$(8.3.6) \qquad \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} t^* = \liminf_{k \to \infty} \left\{\frac{\chi(k) \log^+ |a_k|}{k}\right\}^{\rho_o+1}$$

where $\chi(t)$ is defined as in (8.2.4).

PROOF. Proceeding as in Theorem 6.4 it can be seen that $\psi(k) > \psi(k-1)$ for infinitely many values of k and $\psi(k) \to 1$ as $k \to \infty$. When $\psi(k) > \psi(k-1)$, then term $a_k z^k$ becomes maximum term and we have,

$$\mu(r) = |a_k|\, r^k,\ \nu(r) = k \text{ for } \psi(k-1) \le r < \psi(k).$$

First, let $0 < t^* < \infty$. In view of Lemma 8.5, for given $\varepsilon > 0$, we have for all r such that $0 < r_0 < r < 1$,

$$\log \mu(r) > (t^*-\varepsilon)\ (1-r)^{-\rho_0(r)}.$$

Let $a_{k_1} z^{k_1}$ and $a_{k_2} z^{k_2}$ be two consecutive maximum terms so that $k_1 \le k_2 - 1$. Then

$$\log |a_{k_2}| + k_2 \log r > (t^*-\varepsilon)\ (1-r)^{-\rho_0(r)}.$$

Let $k_1 \le k \le k_2-1$. It is easily seen that

$$\psi(k_1) = \psi(k_1+1) = \cdots = \psi(k) = \cdots = \psi(k_2-1)$$

and

$$|a_k| r^k = |a_{k_2}|\ r^{k_2} \quad \text{for} \quad r = \psi(k).$$

Therefore,

$$\log^+ |a_k| + k \log \psi(k) > (t^*-\varepsilon)\ \{1-\psi(k)\}^{-\rho_0(\psi(k))}.$$

Since $-\log x \ge 1-x$, we get

$$\text{(8.3.7)} \qquad \frac{\chi(k) \log^+ |a_k|}{k} > \frac{(t^*-\varepsilon)\ \chi(k)}{k}\ [\{1-\psi(k)\}^{-\rho_0(\psi(k))} + \frac{k}{t^*-\varepsilon}(1-\psi(k))].$$

Let,

$$S(r) = (1-r)^{-\rho_0(r)} + \frac{k}{t^*-\varepsilon}(1-r).$$

Minimum value of the function $S(r)$ occurs at a point r which is the root of the equation

$$(1-r)^{-\rho_o(r)-1} = \frac{k}{(t^*-\varepsilon)\ (\rho_o+o(1))} .$$

By (8.2.4), this implies,

$$(1-r)^{-1} = \chi\left(\frac{k}{(t^*-\varepsilon)\ (\rho_o+o(1))}\right).$$

Hence,

$$\{1-\psi(k)\}^{-\rho_o(\psi(k))} + \frac{k}{t^*-\varepsilon}\,(1-\psi(k)) \geq \min_{0<r<1} S(r) ,$$

$$\min_{0<r<1} S(r) = \left[\frac{k}{(t^*-\varepsilon)(\rho_o+o(1))\chi(k/(t^*-\varepsilon)(\rho_o+o(1)))} + \frac{k}{(t^*-\varepsilon)\chi(k/(t^*-\varepsilon)(\rho_o+o(1)))}\right]$$

$$= \frac{k}{(t^*-\varepsilon)\chi(k/(t^*-\varepsilon)(\rho_o+o(1)))}\left(\frac{1+\rho_o+o(1)}{\rho_o+o(1)}\right) .$$

Therefore, by (8.3.7) and Lemma 8.4.,

$$\frac{\chi(k)\ \log^+|a_k|}{k} > \frac{1+\rho_o+o(1)}{\rho_o+o(1)}\ \frac{\chi(k)}{\chi(k/(t^*-\varepsilon)(\rho_o+o(1)))}$$

$$= \frac{1+\rho_o+o(1)}{\rho_o+o(1)}\ \{(t^*-\varepsilon)(\rho_o+o(1))\}^{1/(\rho_o+1)} ,$$

which gives

$$(8.3.8) \qquad \liminf_{k\to\infty} \left(\frac{\chi(k)\ \log^+|a_k|}{k}\right)^{\rho_o+1} \geq \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\ t^* .$$

(8.3.8) holds obviously if $t^* = 0$.

We now prove that strict inequality cannot hold in (8.3.8). For, if it holds, then there exists a number $\delta > t^*$ such that

$$\liminf_{k \to \infty} \left(\frac{\chi(k) \log^+ |a_k|}{k} \right)^{\rho_0+1} = \frac{(\rho_0+1)^{\rho_0+1}}{\rho_0^{\rho_0}} - \delta .$$

Let δ_1 be such that $\delta > \delta_1 > t^*$, then for all $k > k_0$,

$$\log^+ |a_k| > \frac{k}{\chi(k)} \frac{\rho_0+1}{\rho_0^{\rho_0/(\rho_0+1)}} \delta_1^{1/(\rho_0+1)}$$

Therefore, using Lemma 8.4, we have for sufficiently large k and for r sufficiently close to 1,

$$\log M(r) > \frac{k}{\chi(k)} \frac{\rho_0+1}{\rho_0^{\rho_0/(\rho_0+1)}} \delta_1^{1/(\rho_0+1)} + k \log r$$

$$\simeq \frac{k}{\chi(k)} \frac{\rho_0+1}{\rho_0^{\rho_0/(\rho_0+1)}} \delta_1^{1/(\rho_0+1)} - k(1-r).$$

Let,

$$k = [\delta_1 \rho_0 (1-r)^{-\rho_0(r)}].$$

Then, in view of (8.2.4), we have asymptotically

$$\log M(r) > \frac{k}{\rho_0 \chi(k/\delta_1 \rho_0)} = \delta_1 (1-r)^{-\rho_0(r)} .$$

Dividing by $(1-r)^{-\rho_0(r)}$ and proceeding to limit, this gives $t^* \geq \delta_1$ which is a contradiction. Hence the proof of the theorem is complete.

COROLLARY. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *belong to the class* U , *have the order* $\rho_o (0 < \rho_o < \infty)$ *and lower type* $t_o (0 \le t_o < \infty)$ *such that* $\psi(k) \equiv |a_k/a_{k+1}|$ *is a nondecreasing function of* k *for* $k > k_o$, *then*

$$\frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}} t_o = \liminf_{k \to \infty} k(\log^+ |a_k|^{1/k})^{\rho_o+1}.$$

This corollary also follows from Theorem 7.4 by taking $\lambda_k = k$.

8.4. In this section we find some inequalities which when applied to functions of the class U give a unified approach to correlate the growth of maximum term with its rank, the growth of geometric mean values with the distribution of its zeros etc., via a D-proximate order. For this purpose we consider a real valued, positive, nondecreasing function $\phi(r)$, $(0 < r < 1)$, which is finite on every subinterval of the open interval $(0,1)$ and tends to infinity as $r \to 1$, and define another function $\psi(r)$ as

(8.4.1) $$\psi(r) = \psi(r_o) + \int_{r_o}^{r} t^{-1} \phi(t)\, dt \quad (0 < r_o < r < 1).$$

Given a D-proximate order $\rho_o(r)$, we define

(8.4.2) $$\lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \psi(r)}{-\log (1-r)} = \begin{matrix} P \\ p \end{matrix}$$

(8.4.3) $$\lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\psi(r)}{(1-r)^{-\rho_o(r)}} = \begin{matrix} D \\ d \end{matrix}$$

(8.4.4)
$$\lim_{r\to 1}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\phi(r)}{r(1-r)^{-\rho_o(r)-1}} = \begin{matrix}Q\\ q\end{matrix}$$

and

(8.4.5)
$$\lim_{r\to 1}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{(\frac{1-r}{r})\ \phi(r)}{\psi(r)} = \begin{matrix}B\\ b\end{matrix}.$$

We now obtain some inequalities involving these constants. In the next section we apply these inequalities for the functions of the class U to find analogues of some well known results for entire functions.

Unless otherwise stated we shall assume that all the constants defined by (8.4.2) to (8.4.5) are nonzero finite.

THEOREM 8.4. *Let the constants* P *and* p, Q *and* q *and* B *and* b *be defined by* (8.4.2), (8.4.4) *and* (8.4.5) *respectively and let the D-proximate order* $\rho_o(r) \to \rho_o\ (0 < \rho_o < \infty)$ *as* $r \to 1$. *Then*

(8.4.6)
$$b \le p \le P \le B$$

and

(8.4.7)
$$q/Q\rho_o \le 1/B \le 1/b \le Q/q\rho_o.$$

PROOF. By (8.4.5) we have for every $\varepsilon > 0$ and for all r such that $1 > r > r_o = r_o(\varepsilon) > 0$,

(8.4.8)
$$b - \varepsilon < \frac{(\frac{1-r}{r})\ \phi(r)}{\psi(r)} < B + \varepsilon.$$

By (8.4.1), we get $\psi'(r) = \frac{\phi(r)}{r}$ for almost all values of r for $0 < r < 1$. Therefore, by (8.4.8)

$$\frac{b-\varepsilon}{1-r} < \frac{\psi'(r)}{\psi(r)} < \frac{B+\varepsilon}{1-r}.$$

Now integrating the above inequalities from r_o to r, we get

$$O(1) - (b-\varepsilon)\log(1-r) < \log \psi(r) < -(B+\varepsilon)\log(1-r) + O(1).$$

Dividing by $-\log(1-r)$ and proceeding to limits as $r \to 1$, (8.4.6) follows.

To prove (8.4.7), for any $\varepsilon > 0$, (8.4.4) gives for all r such that $1 > r > r_o = r_o(\varepsilon) > 0$,

(8.4.9) $$q - \varepsilon < \frac{\phi(r)}{r(1-r)^{-\rho_o(r)-1}} < Q + \varepsilon.$$

This, together with (8.4.1) gives

$$\psi(r) < \psi(r_o) + (Q+\varepsilon)\int_{r_o}^{r} (1-t)^{-\rho(t)-1}\, dt.$$

Now applying (8.2.3) with $\xi = -1$, we get

$$\psi(r) < \psi(r_o) + \frac{Q+\varepsilon}{\rho_o}(1-r)^{-\rho_o(r)} + o((1-r)^{-\rho_o(r)}).$$

Therefore, for all r such that $1 > r > r_o > 0$,

$$\frac{\psi(r)}{(\frac{1-r}{r})\,\phi(r)} < o(1) + \left(\frac{Q+\varepsilon}{\rho_o}\right)\frac{r(1-r)^{-\rho_o(r)-1}}{\phi(r)}.$$

Proceeding to limits, the last inequality gives $1/b \leq Q/\rho_o q$. By similar arguments inequality on the left hand side of (8.4.9) gives $1/B \geq q/Q\rho_o$. Hence the theorem.

COROLLARY 1. $B = b$ *implies* $P = p$.

COROLLARY 2. $Q = q$ *implies* $B = b = \rho_o$.

THEOREM 8.5. *Let the constants* d *and* D *be defined by* (8.4.3) *and* q *and* Q *be defined by* (8.4.4). *If the D-proximate order* $\rho_o(r) \to \rho_o$ $(0 < \rho_o < \infty)$ *as* $r \to 1$, *then*

(8.4.10) $$q \le Q \left(\frac{Q\,\rho_o + q\,\rho_o + 1}{Q\,\rho_o + Q}\right)^{\rho_o+1} \le \rho_o D \le Q$$

and

(8.4.11) $$q \le \rho_o d \le \rho_o q \left\{\left(\frac{1+\rho_o}{\rho_o}\right)\left(\frac{Q}{q}\right)^{1/(\rho_o+1)} - 1\right\} \le Q.$$

Further, $0 < q \le Q < \infty$ *if and only if,* $0 < d \le D < \infty$,

PROOF. By (8.4.4) for every $\varepsilon > 0$ and for all r such that $1 > r > r_o = r_o(\varepsilon)$, we have

(8.4.12) $$\phi(r) > (q-\varepsilon)\, r(1-r)^{-\rho_o(r)-1}.$$

Since, for $1 < k < \infty$,

(8.4.13) $$\int_r^{r+\frac{1}{k}(1-r)} \frac{\phi(t)}{t}\, dt \ge \phi(r)\, \frac{1}{k}\, (1-r),$$

therefore, (8.4.1) and (8.4.12) give,

(8.1.14) $$\psi(r+\tfrac{1}{k}(1-r)) = \psi(r_o) + \int_{r_o}^{r} \frac{\phi(t)}{t}\, dt + \int_r^{r+\frac{1}{k}(1-r)} \frac{\phi(t)}{t}\, dt$$

$$> \psi(r_o) + (q-\varepsilon) \int_{r_o}^{r} (1-t)^{-\rho_o(t)-1}\, dt + \phi(r)\, \frac{1}{k}\, (1-r).$$

Using (8.2.3), with $\xi = -1$, we get

$$\psi(r+\frac{1}{k}(1-r)) > \psi(r_o) + (\frac{q-\varepsilon}{\rho_o})(1-r)^{-\rho_o(r)} + o((1-r)^{-\rho_o(r)}) + \phi(r)\,\frac{1}{k}\,(1-r).$$

Dividing the above inequality by $(\frac{k}{k-1})^{\rho_o(r)}\,(1-r)^{-\rho_o(r)}$ and proceeding to limits, we get

(8.4.15) $$D \geq \frac{q}{\rho_o}\,(\frac{k-1}{k})^{\rho_o} + \frac{Q}{k}\,(\frac{k-1}{k})^{\rho_o}$$

and

(8.4.16) $$d \geq \frac{q}{\rho_o}\,(\frac{k-1}{k})^{\rho_o} + \frac{q}{k}\,(\frac{k-1}{k})^{\rho_o}\,.$$

Further, by (8.4.4), for every $\varepsilon > 0$ and for all r such that $1 > r > r_o = r_o(\varepsilon)$,

(8.4.17) $$\phi(r) < (Q+\varepsilon)\,r\,(1-r)^{-\rho_o(r)-1}\,.$$

Since, for $1 < k < \infty$, we have

(8.4.18) $$\int_r^{r+\frac{1}{k}(1-r)} \frac{\phi(t)}{t}\,dt \leq \phi(r+\frac{1}{k}(1-r))\log(1+\frac{1}{k}(\frac{1-r}{r}))$$

$$\leq \phi(r+\frac{1}{k}(1-r))\,\frac{1}{k}(\frac{1-r}{r})\,,$$

therefore, by (8.4.13) and (8.4.17), we have

$$\psi(r+\frac{1}{k}(1-r)) \leq \psi(r_o) + (Q+\varepsilon)\int_{r_o}^{r} (1-t)^{-\rho_o(t)-1}\,dt + \phi(r+\frac{1}{k}(1-r))\,\frac{1}{k}\,(\frac{1-r}{r}).$$

Using (8.2.3), with $\xi = -1$, the last inequality gives

$$\psi(r+\tfrac{1}{k}(1-r)) \le \psi(r_o)+(\tfrac{Q+\varepsilon}{\rho_o})(1-r)^{-\rho_o(r)}+o((1-r)^{-\rho_o(r)}) + \phi(r+\tfrac{1}{k}(1-r))\,\tfrac{1}{k}\,(\tfrac{1-r}{r}).$$

Dividing by $(\frac{k}{k-1})^{\rho_o(r)}(1-r)^{-\rho_o(r)}$ and proceeding to limits this gives

(8.4.19) $$D \le \frac{Q}{\rho_o}\left(\frac{k-1}{k}\right)^{\rho_o} + \frac{Q}{k-1}$$

and

(8.4.20) $$d \le \frac{Q}{\rho_o}\left(\frac{k-1}{k}\right)^{\rho_o} + \frac{q}{k-1}.$$

Since (8.4.16) and (8.4.19) hold for all k such that $1 < k < \infty$, letting $k \to \infty$ in these inequalities, we get $q \le \rho_o d$ and $Q \ge \rho_o D$. Hence, $q = Q$ implies $D = d = Q/\rho_o$ and the theorem is established in this case. Now, let $Q > q$. With $k = \frac{Q(\rho_o+1)}{Q-q}$, (8.4.15) gives

$$D \ge \left(\frac{Q\rho_o + q\rho_o}{Q\rho_o + Q}\right)^{\rho_o}\left(\frac{q}{\rho_o} + \frac{Q-q}{\rho_o+1}\right) = \frac{Q}{\rho_o}\left(\frac{Q\rho_o + q}{Q\rho_o + Q}\right)^{\rho_o+1} \ge \frac{q}{\rho_o}.$$

Putting $\frac{k-1}{k} = (q/Q)^{1/(\rho_o+1)}$, (8.4.20) gives

$$d \le \frac{Q}{\rho_o}\left(\frac{q}{Q}\right)^{\rho_o/(\rho_o+1)} + q\left\{\left(\frac{Q}{q}\right)^{1/(\rho_o+1)} - 1\right\}$$

$$= q\left\{\left(\frac{1+\rho_o}{\rho_o}\right)\left(\frac{Q}{q}\right)^{1/(\rho_o+1)} - 1\right\} \le \frac{Q}{\rho_o}.$$

This completes the proof of (8.4.10) and (8.4.11). To prove the remaining part of the theorem, we observe that if $q > 0$, then by (8.4.16), $d > 0$, and if $Q < \infty$, then by (8.4.19), $D < \infty$. Also (8.4.15) shows that $D < \infty$ implies $Q < \infty$. Further, if $d > 0$, then $q > 0$; because if

$q = 0$, then by (8.4.20), $d \leq \frac{Q}{\rho_0} \left(\frac{k-1}{k}\right)^{\rho_0}$. Since this holds for all k such that $1 < k < \infty$, making $k \to 1$, we would have $d = 0$. Hence the theorem.

It follows from (8.4.10) and (8.4.11) that

$$(8.4.21) \qquad (a) \quad Q \leq \frac{(\rho_0+1)^{\rho_0+1}}{\rho_0^{\rho_0}} D\ ; \quad (b) \quad q \leq \rho_0 D$$

and

$$(8.4.22) \qquad q \leq \rho_0\, d \leq \rho_0\, D \leq Q.$$

By (8.4.22), it follows that $q = Q$ implies $D = d$. However converse of this is also true and we have

THEOREM 8.6. *Let $\psi(r)$ and Q be defined by (8.4.1) and (8.4.4) and let the D-proximate order $\rho_0(r) \to \rho_0 (0 < \rho_0 < \infty)$ as $r \to 1$. Then*

$$(8.4.23) \qquad \phi(r) \sim Q\, r(1-r)^{-\rho_0(r)-1} \quad \text{as } r \to 1\ ,$$

if and only if,

$$(8.4.24) \qquad \psi(r) \sim \frac{Q}{\rho_0} (1-r)^{-\rho_0(r)} \quad \text{as } r \to 1.$$

PROOF. First suppose that $\phi(r) \sim Qr(1-r)^{-\rho_0(r)-1}$ as $r \to 1$. Then, it follows immediately from (8.4.22) that $D = d = Q/\rho_0$ and consequently (8.4.24) holds.

Next, (8.4.24) implies $D = d = Q/\rho_0$. Hence $\psi(r) \sim D(1-r)^{-\rho_0(r)}$ as $r \to 1$. Therefore, by (8.4.1), for $1 < k < \infty$, we have

$$(8.4.25) \qquad \int_{r}^{r+\frac{1}{k}(1-r)} \frac{\phi(t)}{t}\, dt = \psi(r+\frac{1}{k}(1-r)) - \psi(r)$$

$$= (1-r)^{-\rho_o(r)} \{(\frac{k}{k-1})^{\rho_o(r)} - 1\} D + o((1-r)^{-\rho_o(r)}).$$

Now (8.4.25) and (8.4.13) give asymptotically

$$\frac{\phi(r)}{r(1-r)^{-\rho_o(r)-1}} \leq \frac{k\, D}{r} \{(\frac{k}{k-1})^{\rho_o(r)} - 1\}.$$

Therefore,

$$\limsup_{r \to 1} \frac{\phi(r)}{r(1-r)^{-\rho_o(r)-1}} \leq k \{(\frac{k}{k-1})^{\rho_o} - 1\} D.$$

Letting $k \to \infty$ on the right hand side of the above inequality, we get

$$(8.4.26) \qquad Q \leq \rho_o D.$$

Again, by (8.4.25) and (8.4.18), we get asymptotically

$$\phi(r + \frac{1}{k}(1-r)) \frac{1}{k} (\frac{1-r}{r}) > (1-r)^{-\rho_o(r)} \{(\frac{k}{k-1})^{\rho_o(r)} - 1\} D.$$

Dividing this inequality by $(1-r)^{-\rho_o(r)} \frac{1}{k} (\frac{k}{k-1})^{\rho_o + 1}$ and proceeding to limits, we have

$$\liminf_{r \to 1} \frac{\phi(r)}{r(1-r)^{-\rho_o(r)-1}} \geq \frac{k\{(\frac{k}{k-1})^{\rho_o} - 1\}}{(\frac{k}{k-1})^{\rho_o+1}} D.$$

Letting $k \to \infty$,

$$(8.4.27) \qquad q \geq \rho_o D.$$

Since $q \le Q$, (8.4.26) and (8.4.27) imply $q = Q = \rho_0 D$ and consequently (8.4.23) holds.

REMARK. As a consequence of Theorem 8.6 we get that if either (8.4.23) or (8.4.24) holds then

(8.4.28) $$B = b = \rho_0 \, .$$

Next we have

THEOREM 8.7. *Let the constants* d *and* D *be defined by* (8.4.3) *and* q *and* Q *be defined by* (8.4.4). *If the D-proximate order* $\rho_0(r) \to \rho_0 (0 < \rho_0 < \infty)$ *as* $r \to 1$, *then*

(8.4.29) $$Q + q + d \le Q + (\rho_0+1)\, d \le \frac{(\rho_0+1)^{\rho_0+1}}{\rho_0^{\rho_0}}\, D$$

and

(8.4.30) $$\frac{(\rho_0+1)^{\rho_0}}{\rho_0^{\rho_0+1}}\, q + D \ge \left(\frac{\rho_0+1}{\rho_0}\right)^{\rho_0} d.$$

PROOF. (8.4.1) and (8.4.13) with $k = \rho_0 + 1$ give

$$\frac{r}{\rho_0+1}\, \frac{\phi(r)}{r(1-r)^{-\rho_0(r)-1}} + \frac{\psi(r)}{(1-r)^{-\rho_0(r)}} \le \frac{\psi(r+\frac{1-r}{1+\rho_0})}{(1-r)^{-\rho_0(r)}} \, .$$

Proceeding to limits, this gives

$$Q + (\rho_0+1)\, d \le \frac{(\rho_0+1)^{\rho_0+1}}{\rho_0^{\rho_0}}\, D.$$

Since $q \le \rho_0 d$, (8.4.29) follows.

Next by (8.4.1) and (8.4.18), with $k = \rho_0 + 1$, we have

$$\frac{1}{\rho_o+1}\ \frac{\phi(r+\frac{1-r}{\rho_o+1})}{r(1-r)^{-1}} + \psi(r) \geq \psi(r+\frac{1-r}{\rho_o+1}).$$

Dividing by $(1-r)^{-\rho_o(r)}$ and proceeding to limits we get (8.4.30) and the theorem is completely established.

REMARK. If equality holds in (8.4.21)(a), then by (8.4.29), $q = d = 0$. Moreover if equality holds in (8.4.21) (b), then $q = Q$ since in that case by (8.4.10), $\left(\frac{Q\ \rho_o+q}{Q\ \rho_o+Q}\right)^{\rho_o+1} = \frac{q}{Q}$ which is possible, if and only if, $q = Q$.

8.5. We now give some applications of the results obtained in the previous section.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in the unit disc having order $\rho_o(0 < \rho_o < \infty)$, type T_o and lower type t_o. Let $\mu(r)$ be the maximum term and $\nu(r)$ be the central index of $f(z)$.

(i) Let $\phi(r) = \nu(r)$, then by (1.6.1), $\psi(r)$ becomes $\log \mu(r)$. Let the order and lower order of $f(z)$ be ρ_o and λ_o respectively. If we take $\rho_o(r) = \rho_o$, then inequalities (8.4.6) and (8.4.7) remain true with $P = \rho_o$, $p = \lambda_o$ and Q and q as its growth number and lower growth number respectively. Corollary 1 implies that if $\log \mu(r) \sim (1-r)\ \nu(r)$ as $r \to 1$ then $\rho_o = \lambda_o$ i.e. $f(z)$ is of regular growth. Similar results for the case of entire functions were obtained by Shah [61], [68].

(ii) Let $n(r)$ denote the number of zeros of $f(z)$ in $|z| \leq r$, $0 < r < 1$. Setting $\phi(r) = n(r)$, $\psi(r)$ becomes $N(r) = \int_0^r \frac{n(t)}{t}\,dt$. Taking $\rho_o(r) = \rho_o$, from (8.4.7), we get the inequalities

$$\frac{\ell}{\rho_o L} \le \frac{1}{B} \le \frac{1}{b} \le \frac{L}{\rho_o \ell},$$

where

(8.5.1)
$$\begin{matrix} L \\ \ell \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{n(r)}{r(1-r)^{-\rho_o - 1}},$$

and

$$\begin{matrix} B \\ b \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\left(\frac{1-r}{r}\right) n(r)}{N(r)}.$$

A corresponding result for the case of entire functions is due to Juneja [32] .

(iii) Let $\phi(r) = \nu(r)$ and $\rho_o(r) = \rho_o$. Then by (1.6.1) $\psi(r)$ becomes $\log \mu(r)$. Hence, in view of Lemma 8.5, Theorem 8.5 remains valid for the function $f(z) \varepsilon\, \mathcal{U}$ with D and d replaced by its type T_o and lower type t_o and Q and q replaced by its growth number μ_o and lower growth number δ_o . The analogous inequalities for the class of entire functions are well known [66] .

Geometric mean value $G(r)$ of a function $f(z) \varepsilon\, \mathcal{U}$ is defined as

$$\log G(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta, \; |z| = r \quad (0 < r < 1).$$

By Jensen's theorem

$$\log G(r) = \int_0^r \frac{n(t)}{t} dt - \log |f(0)|, \; f(0) \neq 0$$

$n(t)$ being the number of zeros of $f(z)$ in $|z| \le t$, $0 < t < 1$. Hence

(iv) Putting $\phi(r) = n(r)$ and $\rho_0(r) = \rho_0$, we get $\psi(r) \sim \log G(r)$ as $r \to 1$. Thus, Theorem 8.5 gives the estimation of the growth of the geometric mean value of a function of the class U, with respect to its number of zeros in $|z| \leq r$ $(0 < r < 1)$. A similar result in a particular form for the class of entire functions is proved by R.P. Srivastav [79] and S.N. Srivastava [86] .

Let $A(r) = \max\limits_{|z| = r} \{\text{Re } f(z)\}$, $0 < r < 1$. Then since [87, p 86] $A(r) \leq M(r)$ and $|a_n| r^n \leq \max\{4A(r), 0\} - 2 \text{ Re } f(0)$, for all values of n and r $(0 < r < 1)$, we have by Lemma 8.5 that

$$\begin{matrix} T^* \\ t^* \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log A(r)}{(1-r)^{-\rho_0(r)}} .$$

Now since $\log A(r)$ is a convex function of $\log r (0 < r < 1)$, we can write

$$\log A(r) = \log A(r_0) + \int_{r_0}^{r} \frac{a(t)}{t} dt$$

where $a(t)$ is an indefinitely increasing function of t $(0 < t < 1)$ and is continuous in adjacent intervals of $(0,1)$.

(v) Let $\phi(r) = a(r)$ and $\rho_0(r) = \rho_0$. Then, Theorem 8.5 gives results analogous to those of Jain [30] obtained for the class of entire functions.

(vi) Let $f(0) \neq 0$, $\phi(r) = n(r)$ and $\rho_0(r) = \rho_0$. Then,

$$\psi(r) = N(r) = \int_0^r \frac{n(t)}{t} dt.$$

By Jensen's theorem,

$$N(r) = \int_0^r \frac{n(t)}{t} dt \leq \log M(r) \quad (0 < r < 1).$$

Hence, (8.4.21) is valid for the function $f(z) \varepsilon\, \mathcal{U}$, with D replaced by its type T_o and Q and q replaced by the quantities L and ℓ defined by (8.5.1). An analogous result is well known [12] for the class of entire functions.

Since $\log M(r)$ is a convex function of $\log r$ we have the relation

$$\log M(r) = \log M(r_o) + \int_{r_o}^{r} \frac{w(t)}{t}\, dt, \quad (0 < r_o < r < 1)$$

where $w(t)$ is an indefinitely increasing function of t $(0 < t < 1)$, continuous in the adjacent subintervals of $(0,1)$. Thus, $r \dfrac{M'(r)}{M(r)} = w(r)$, for almost all values of $r (0 < r < 1)$, $M'(r)$ being the derivative of $M(r)$, wherever it exists.

(vii) Let $\phi(r) = w(r)$ and $\rho_o(r) = \rho_o$. Then (8.4.21)(a) gives

$$\limsup_{r \to 1} \frac{M'(r)}{M(r)\,(1-r)^{-\rho_o - 1}} \leq \frac{(\rho_o+1)^{\rho_o+1}}{\rho_o^{\rho_o}}\, T_o$$

where T_o is the type of $f(z)$. An analogous result for the class of entire functions is due to Kövari [41].

(viii) Let $\phi(r) = \nu(r)$ and $\rho_o(r) = \rho_o$. Then, Theorem 8.6 gives the asymptotic behaviour of $\log \mu(r)$ and $\nu(r)$ for the functions of the class $\mathcal{U}$. A similar result for the class of entire functions was proved by Shah [70]. Taking $\phi(r) = n(r)$ and applying Jensen's theorem, Theorem 8.6 also gives asymptotic relation between the geometric mean value and number of zeros of $f(z)$ in $|z| \leq r$ $(0 < r < 1)$. Analogous results for the class of entire functions are proved by Srivastava [86].

(ix) Let $\phi(r) = \nu(r)$ and $\rho_o(r) = \rho_o$. Then, Theorem 8.7 remains valid for the function $f(z) \varepsilon\, \mathcal{U}$ with D and d replaced by its type T_o and the lower type t_o and Q and q replaced by its growth number μ_o and lower growth

number δ_0 . We can also take $\phi(r) = n(r)$ and $\rho_0(r) = \rho_0$ so that in view of Jensen's theorem, inequalities (8.4.29) and (8.4.30) give relations between the geometric mean value of the function $f(z) \in U$ and the number of its zeros in $|z| \leq r$ $(0 < r < 1)$.

(x) In fact, $\psi(r)$ can be replaced by any convex function of log r associated with the function $f(z) \in U$ and its growth can be measured in term of the corresponding function $\phi(r)$ occuring under the integral sign in (8.5.1). Thus $\psi(r)$ could be replaced by its mean values $\log \mu_\delta(r)$, where

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \qquad (1 \leq \delta < \infty)$$

or by its weighted mean values $\log m_{\delta,k}(r)$, where

$$m_{\delta,k}(r) = \frac{k+1}{r^{k+1}} \int_0^r \mu_\delta(x)\, x^k \, dx\ , \quad (1 \leq k < \infty)$$

or by its characteristic function $m(r)$ defined by

$$m(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

or by its weighted geometric mean values $\log g_\delta(r)$, where

$$\log g_\delta(r) = \frac{\delta+1}{r^{\delta+1}} \int_0^r x^\delta \log G(x)\, dx\ , \quad (1 \leq \delta < \infty),$$

and so on.

APPENDIX

Here we obtain coefficient characterizations for the lower order of an entire Dirichlet series.

Let $f(s)$ be an entire function of the complex variable $s = \sigma + i\,t$ defined by the everywhere absolutely convergent Dirichlet series

$$(1) \qquad \sum_{n=1}^{\infty} a_n \exp(s\,\lambda_n)$$

where $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots,\ \lambda_n \to \infty$ as $n \to \infty$.

Let

$$M(\sigma) = \max_{-\infty<t<\infty} |f(\sigma+i\,t)|.$$

The Ritt order ρ^* and lower order λ^* ($0 \le \lambda^* \le \rho^* \le \infty$) of $f(s)$ are defined by

$$(2) \qquad \begin{matrix}\rho^* \\ \\ \lambda^*\end{matrix} = \lim_{\sigma \to \infty} \begin{matrix}\sup \\ \\ \inf\end{matrix} \frac{\log \log M(\sigma)}{\sigma}.$$

Azpeitia [5] has shown that if

$$(3) \qquad \lim_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log n} = \infty$$

then the Ritt order ρ^* of $f(s)$ is given by

$$(4) \qquad \rho^* = \limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}.$$

A similar result for the lower order λ^* does not always hold. In fact, it has been shown [37] that

(5) $$\lambda^* \geq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}}$$

and that if

(6) $$\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

(7) $\log |a_n/a_{n+1}|/(\lambda_{n+1} - \lambda_n)$ forms a nondecreasing function of n for $n > n_o$,

then

(8) $$\lambda^* = \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} .$$

We now obtain formulae for the lower order which hold for every entire Dirichlet series satisfying (6). We shall suppose throughout that $f(s)$ is an entire function defined by everywhere absolutely convergent Dirichlet series (1).

We have the following theorem.

THEOREM. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of lower order $\lambda^*(0 \leq \lambda^* \leq \infty)$ such that (6) is satisfied, then

(9) $$\lambda^* = \max_{\{m_k\}} \left[\liminf_{k \to \infty} \frac{\lambda_{m_k} \log \lambda_{m_{k-1}}}{\log |a_{m_k}|^{-1}}\right]$$

(10) $$= \max_{\{m_k\}} \left[\liminf_{k \to \infty} \frac{(\lambda_{m_k} - \lambda_{m_{k-1}}) \log \lambda_{m_{k-1}}}{\log |a_{m_{k-1}}/a_{m_k}|}\right].$$

Maximum in (9) and (10) is taken over all increasing sequences $\{m_k\}$ of natural numbers.

We first prove two lemmas.

LEMMA 1. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function of lower order* $\lambda^*(0 \le \lambda^* \le \infty)$ *and let* $\{m_k\}$ *be an increasing sequence of positive integers, then*

$$(11) \qquad \lambda^* \ge \liminf_{k \to \infty} \frac{\lambda_{m_k} \log \lambda_{m_{k-1}}}{\log |a_{m_k}|^{-1}} .$$

PROOF. Let lim inf in (11) be denoted by α, then $0 \le \alpha \le \infty$. First suppose that $0 < \alpha < \infty$. For any ε such that $\alpha > \varepsilon > 0$, we can choose a fixed integer $N = N(\varepsilon)$ such that for all $m \ge N$, we have

$$(12) \qquad \log |a_{m_k}| > - \frac{\lambda_{m_k} \log \lambda_{m_{k-1}}}{\alpha-\varepsilon} .$$

Let $\exp(\sigma_k - 1) = \lambda_{m_{k-1}}^{1/(\alpha-\varepsilon)}$ for $k = 1,2,3,\ldots$. If $k > N$, $\sigma_k \le \sigma \le \sigma_{k+1}$, then by (12)

$$\begin{aligned} \log M(\sigma) &\ge \log |a_{m_k}| + \sigma_k \lambda_{m_k} \\ &> - \frac{\lambda_{m_k} \log \lambda_{m_{k-1}}}{\alpha-\varepsilon} + \lambda_{m_k} \log \{e\, \lambda_{m_{k-1}}^{1/(\alpha-\varepsilon)}\} \\ &= \lambda_{m_k} \\ &= \exp\{(\sigma_{k+1} - 1)(\alpha-\varepsilon)\}. \end{aligned}$$

So,

$$\begin{aligned} \log\log M(\sigma) &\ge (\sigma_{k+1} - 1)(\alpha-\varepsilon) \\ &\ge (\sigma-1)(\alpha-\varepsilon) \end{aligned}$$

or,

$$\lambda^* \equiv \liminf_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} \geq \alpha .$$

This inequality is obviously true if $\alpha = 0$. If $\alpha = \infty$, the above arguments with an arbitrarily large number in place of $(\alpha - \varepsilon)$ give $\lambda^* = \infty$. This completes the proof of the lemma.

LEMMA 2. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of lower order $\lambda^* (0 \leq \lambda^* < \infty)$ and let $\{m_k\}$ be an increasing sequence of natural numbers, then

$$\lambda^* \geq \liminf_{k \to \infty} \frac{(\lambda_{m_k} - \lambda_{m_{k-1}}) \log \lambda_{m_{k-1}}}{\log |a_{m_{k-1}} / a_{m_k}|} . \tag{13}$$

The proof of this lemma is similar to that of Lemma 1 (see, e.g., [36]) hence we omit it.

PROOF OF THE THEOREM. Let $\mu(\sigma)$ denote the maximum term of the series $\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ for Re $s = \sigma$, i.e., $\mu(\sigma) = \max_{n \geq 0} \{|a_n| \exp(\sigma\lambda_n)\}$ and let $\nu(\sigma) = \max \{n : \mu(\sigma) = |a_n| \exp(\sigma \lambda_n)\}$. Let the range of $\nu(\sigma)$ be $\{n_k\}$. We shall denote the jump points of $\nu(\sigma)$ by $\rho_n = \rho(n)$. We limit ourselves to nontrivial cases by supposing that $n_k \to \infty$ as $k \to \infty$ so that $\rho(n) \to \infty$ as $n \to \infty$. We thus have

$$\rho(n) \leq \rho(n+1), \ n = 1,2,\ldots$$

$$\rho(n_k) < \rho(n_k+1) = \ldots = \rho(n_{k+1}), \ k = 1,2,\ldots,$$

$$\lambda_{\nu(\sigma)} = \lambda_{n_k} \text{ when } \rho(n_k) \leq \sigma < \rho(n_{k+1}), k = 1,2,\ldots .$$

Also since $|a_{n_k}| \exp(\sigma\lambda_{n_k})$ and $|a_{n_{k+1}}| \exp(\sigma\lambda_{n_{k+1}})$ are the two consecutive maximum terms, we have

$$(14) \qquad \rho(n_k) = \frac{\log |a_{n_{k-1}}/a_{n_k}|}{\lambda_{n_k} - \lambda_{n_{k-1}}} .$$

$\rho(n_k)$ is an increasing function of k. Further, if f(s) satisfies (6) we have [37] .

$$(15) \qquad \lambda^* = \liminf_{\sigma \to \infty} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} .$$

Now consider the entire function $g(s) = \sum_{k=1}^{\infty} a_{n_k} \exp(s\lambda_{n_k})$, where $\{n_k\}$ denotes the range of $\nu(\sigma)$. Since f(s) satisfies (6), g(s) also satisfies the same condition. Further, for any σ, f(s) and g(s) have the same maximum term, hence the order and lower order of g(s) are the same as those of f(s). Thus, g(s) is of lower order λ^* and since $\rho(n_k)$ is an increasing function of k, g(s) satisfies (7). Hence, by (8)

$$(16) \qquad \lambda^* = \liminf_{k \to \infty} \frac{\lambda_{n_k} \log \lambda_{n_{k-1}}}{\log |a_{n_k}|^{-1}} .$$

But, from Lemma 1, we have

$$(17) \qquad \lambda^* \geq \max_{\{m_k\}} \liminf_{k \to \infty} \frac{\lambda_{m_k} \log \lambda_{m_{k-1}}}{\log |a_{m_k}|^{-1}} .$$

Comparing (16) and (17), we get (9).

To prove (10), if $c = \min\{1, \frac{\rho(n_{k+1}) - \rho(n_k)}{2}\}$ by using (15) we have, for $n_k \leq n \leq n_{k+1}$,

$$\lambda^* = \liminf_{n \to \infty} \frac{\log \lambda_n}{\rho(n)} \geq \liminf_{k \to \infty} \frac{\log \lambda_{n_k}}{\rho(n_{k+1})} = \liminf_{k \to \infty} \frac{\log \lambda_{n_k}}{\rho(n_{k+1}) - c} \geq \lambda^* .$$

Thus,

$$\lambda^* = \liminf_{k \to \infty} \frac{\log \lambda_{n_k}}{\rho(n_{k+1})} .$$

Substituting the value of $\rho(n_{k+1})$ from (14), we get

$$\lambda^* = \liminf_{k \to \infty} \frac{(\lambda_{n_{k+1}} - \lambda_{n_k}) \log \lambda_{n_k}}{\log |a_{n_k}/a_{n_{k+1}}|} . \tag{18}$$

But, from Lemma 2, we have

$$\lambda^* \geq \max_{\{m_k\}} \liminf_{k \to \infty} \frac{(\lambda_{m_k} - \lambda_{m_{k-1}}) \log \lambda_{m_{k-1}}}{\log |a_{m_{k-1}}/a_{m_k}|} . \tag{19}$$

Comparing (18) and (19), we get (10). Hence the theorem.

COROLLARY 1. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function of lower order* $\lambda^*(0 \leq \lambda^* \leq \infty)$ *such that* (6) *is satisfied and* $\psi(n) \equiv \log |a_n/a_{n+1}|/(\lambda_{n+1} - \lambda_n)$ *forms a nondecreasing function of* n *for* $n > n_0$, *then*

$$\lambda^* = \liminf_{n \to \infty} \frac{(\lambda_n - \lambda_{n-1}) \log \lambda_{n-1}}{\log |a_{n-1}/a_n|} .$$

COROLLARY 2. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function of Ritt order* $\rho^*(0 \leq \rho^* \leq \infty)$ *and lower order* $\lambda^*(0 \leq \lambda^* \leq \infty)$ *such that* (6) *is satisfied. Then,*

$$\lambda^* \leq \rho^* \liminf_{k \to \infty} \frac{\log \lambda_{k-1}}{\log \lambda_k} .$$

LIST OF RESEARCH PAPERS

1. On the lower order of entire functions.
   J. London Math. Soc. (2). To appear.

2. Coefficient characterizations of entire functions.
   Communicated for publication.

3. On the coefficients **of an** entire series.
   Presented at the annual meeting of Bharat Ganita Parishad, Lucknow, 1971.

4. On the lower order of functions analytic in the unit disc.
   Math. Japon, 17 (1). To appear.

5. On the lower order of functions analytic in the unit disc. II.
   Communicated for publication.

6. On extreme rates of growth of functions analytic in a finite disc.
   Presented at the annual meeting of Bharat Ganita Parishad, Lucknow, 1972. Abstract appearing in Notices Amer. Math. Soc. August, 1972.

7. On the growth of functions analytic in the unit disc.
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8. A note on the proximate order of functions analytic in the unit disc.
   Istanbul Univ. Fen. Fak . Mec. Ser. A. 36 (1971). To appear

9. On the proximate order and maximum term of functions analytic in the unit disc.
   Communicated for publication.

10. On some inequalities related to functions analytic in the unit disc.
    Communicated for publication.

11. On the lower order of an entire Dirichlet series.
    Math. Ann. 197 (1972), 128-132. To appear.

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